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# Motives and admissible representations of automorphism groups of fields

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**Abstract** Some of basic properties of the groups of automorphisms of algebraically closed fields and of their smooth representations are studied. In characteristic zero, Grothendieck motives modulo numerical equivalence are identified with a full subcategory in the category of graded smooth representations of certain automorphism groups of algebraically closed fields.

## Contents



## 1 Introduction

Let $k$ be an algebraically closed field of characteristic zero. (This will be the case through out the paper, except Appendix B, where all results of §2 are extended to positive characteristic.) Let $F$ be an algebraically closed extension of $k$ of transcendence degree $n$, $1 \leq n \leq \infty$, and let $G = G_{F/k}$ be the group of automorphisms over $k$ of the field $F$. Let the set of subgroups

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$U_{k(x)} := \mathrm{Aut}(F/k(x))$ for all $x \in F$ be a base of neighbourhoods of the identity in $G$.

This paper arose from an attempt to (i) compare properties of various "geometric" categories with properties of various categories of smooth (i.e., with open stabilizers) representations of $G$, and (ii) to find analogues for the group $G$ of familiar results of representation theory of $p$-adic groups.

The group $G$ is very big, in particular, it contains the groups $\mathrm{Aut}(L/k)$ as its sub-quotients for all sub-extensions $k \subset L \subset F$. All reduced irreducible algebraic groups of dimension $\le n$, the group $\mathrm{PGL}_{n+1}k$, some adelic groups are subgroups of groups of type $\mathrm{Aut}(L/k)$.

One of the main results of the paper (Theorem 2.9) is the simplicity (in topological sense) of the group $G$ in the case $n = \infty$, and of the subgroup $G^\circ$ of $G$ generated by its compact subgroups in general. This is also true in positive characteristic, and implies (Corollary 2.11) that in the case $n = \infty$ any non-trivial continuous representation of $G$ is faithful; and in the case $n < \infty$ any non-faithful continuous representation of $G$ factors through a discrete quotient $G/G^\circ$ of $G$. Another consequence is that $G^\circ$ (and $G$ if $n = \infty$) admits no smooth representations of finite degree.

Unfortunately, I do not know much about the group $G/G^\circ$, and this is one of the reasons why I prefer to work in the "stable" case $n = \infty$.

There is an evident link between representations of $G$ and some geometric objects. Namely, for a scheme $X$ over $k$ there is a natural smooth $G$-action on the group of cycles on $X_F := X \times_k F$. Conversely, any smooth cyclic $G$-module is a quotient of the $G$-module of "generic" 0-cycles on $X_F$ for an appropriate irreducible variety $X$ of dimension $\le n$ over $k$. The Hecke algebras, playing an important rôle in representation theory of locally compact groups, become in our case algebras of non-degenerate correspondences on certain varieties over $k$, cf. §3, p.15.

In some cases one can identify the groups of morphisms between geometric objects with the groups of morphisms between corresponding $G$-modules (cf. Propositions 3.6 and 4.3, and Corollary 3.7).

(Homological) Grothendieck motives are pairs $(X, \pi)$ consisting of a smooth projective variety $X$ over $k$ with irreducible components $X_j$ and a projector $\pi = \pi^2 \in \bigoplus_j A^{\dim X_j}(X_j \times_k X_j)$ in the algebra of correspondences on $X$ modulo an adequate equivalence relation. The morphisms are defined by $\mathrm{Hom}((X', \pi'), (X, \pi)) = \bigoplus_{i,j} \pi_j \cdot A^{\dim X_j}(X_j \times_k X'_i) \cdot \pi'_i$. The category of Grothendieck motives carries an additive and a tensor structures:

$$(X', \pi') \bigoplus (X, \pi) := (X' \coprod X, \pi' \oplus \pi),$$
$$(X', \pi') \otimes (X, \pi) := (X' \times_k X, \pi' \times_k \pi).$$

A *primitive $q$-motive* is a pair $(X, \pi)$ as above with $\dim X = q$ and $\pi \cdot A^q(X \times_k Y \times \mathbb{P}^1) = 0$ for any smooth projective variety $Y$ over $k$ with $\dim Y < q$. For instance, the category of the primitive 1-motives modulo numerical equivalence is equivalent to the category of abelian varieties over $k$ with morphisms tensored with $\mathbb{Q}$. If the adequate equivalence relation is

numerical equivalence, it follows from a result of Jannsen [Jan] that any Grothendieck motive is semi-simple and admits "primitive" decomposition $\bigoplus_{i,j} M_{ij} \otimes \mathbb{L}^{\otimes i}$, where $M_{ij}$ is a primitive $j$-motive and $\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times \{0\})$ (see Remark on p.27).

The major part of the results of Section 3 can be summarized as follows.

**Theorem 1.1.** *1. If $n = \infty$ there is a fully faithful functor $\mathbb{B}^\bullet$:*

$$\left\{\begin{array}{c}\text{motives over } k \text{ modulo}\\ \text{numerical equivalence}\end{array}\right\} \xrightarrow{\mathbb{B}^\bullet} \left\{\begin{array}{c}\text{graded semi-simple admissible}\\ G\text{-modules of finite type}\end{array}\right\}.$$

*The grading corresponds to powers of the motive $\mathbb{L}$ in the "primitive" decomposition above.*

*2. For any $1 \le n \le \infty$ and each $q \ge 0$ there is a functor $\mathfrak{B}^q$, fully faithful if $q \le n$:*

$$\left\{\begin{array}{c}\text{primitive } q\text{-motives over } k\\ \text{modulo numerical equivalence}\end{array}\right\} \xrightarrow{\mathfrak{B}^q} \left\{\begin{array}{c}\text{semi-simple admissible}\\ G\text{-modules of finite type}\\ \text{and of level } q\end{array}\right\}.$$

*(One says that a semi-simple admissible $G$-module $W$ is of level $q$ if $N_qW = W$ and $N_{q-1}W = 0$ for the filtration $N_\bullet$ defined in the beginning of §6.1.)*

*3. If $n < \infty$ then the $G$-module $\mathfrak{B}^n(M)$ carries a bilinear symmetric non-degenerate $G$-equivariant form with values in an oriented $G$-module $\mathbb{Q}(\chi)$ of degree 1, where $M = (X, \Delta_{k(X)})$ is the maximal primitive $n$-submotive of the motive $(X, \Delta_X)$ and $\dim X = n$.*
*This form is definite if, for $(n-1)$-cycles on $2n$-dimensional complex varieties, numerical equivalence coincides with homological (e.g., for $n \le 2$), and therefore, $\mathfrak{B}^n$ factors through the subcategory of "polarizable" $G$-modules (i.e., carrying a positive form as above).*

This is a direct consequence of Corollary 3.7 and Propositions 3.8, 3.19, 3.21. Roughly speaking, the functors $\mathfrak{B}^q$ and $\mathbb{B}^\bullet = \oplus_j^{\text{graded}} \mathbb{B}^{[j]}$ are defined as spaces of 0-cycles defined over $F$ modulo "numerical equivalence over $k$". Details are in §3.2, p.26, where it is shown that they are pro-representable. It follows from Proposition 3.17 that $\mathfrak{B}^q((X, \pi))$ depends only on the birational class of $X$. Moreover, the functor $\mathfrak{B}^1$ of Theorem 1.1(2) is an equivalence of categories when $n = \infty$, cf. §6.2, and by Corollary 4.4, the composition of the functor $\mathfrak{B}^1$ with the forgetful functor to the category of $G^\circ$-modules is also fully faithful.

*Conjecture 1.2.* The functor $\mathbb{B}^\bullet$ is an equivalence of categories. Equivalently, for any $q \ge 0$ the functor $\mathfrak{B}^q$ is an equivalence of categories if $n = \infty$.

The section is concluded by showing (Corollary 3.24) that any polarizable representation (in the sense of Theorem 1.1(3)) is infinite-dimensional. This is deduced from a vanishing result (Corollary 3.23) for representations

of the Hecke algebra of the subgroup $\mathrm{Gal}(F/L(x))$, induced by polarizable representations of $G$ (here $L$ is a subextension of $k$ in $F$ of finite type and $x$ an element of $F$ transcendental over $L$ with $F=\overline{L(x)}$) corresponding to the triviality of the primitive $n$-submotives of $(Y\times\mathbb{P}^1,\pi)$, where $\dim Y<n$.

Corollary 3.23 suggests also that the category of the primitive $n$-motives is not too far from the category of the polarizable $G$-modules (at least, if $n\le 2$). However, as the twists of the polarizables by order-two characters of $G$ are again polarizable, but not "motivic" (cf. Corollary 4.5), one should impose some additional conditions. One of such conditions could be the "stability" in the sense that for any algebraically closed extension $F'$ of $F$ any "motivic" representation is isomorphic to $W^{G_{F'/F}}$ for some smooth $G_{F'/k}$-module $W$; another one could be the "arithmeticity" in the sense that any "motivic" representation admits an extension $W$ by a module of lower level (in the sense of filtration $N_\bullet$) with $W$ isomorphic to the restriction to $G\subseteq G_{F/k'}$ of a smooth $G_{F/k'}$-module for a subfield $k'\subseteq k$ of finite type over $\mathbb{Q}$...

By analogy with the Langlands correspondences, one can call the $G$-modules in the image of $\mathfrak{B}^n$ *cuspidal*. For the groups GL over a local non-archimedian field there are several equivalent definitions of quasicuspidal representations. One of them: all matrix coefficients (the functions $\langle\sigma w,\widetilde{w}\rangle$, cf. p.34) are compactly supported modulo the center. However, it is shown in §4.3 that there are no such representations for any subgroup of $G$ containing $G^\circ$. This is a consequence of the irreducibility of the smooth $G$-modules $F/k$ and $F^\times/k^\times$, considered as modules over the subgroup $G^\circ$ of $G$, and their faithfulness as modules over the algebra of compactly supported measures on $G$ shown in §4.1.

In §5.1, in the case $n=\infty$, various analogues of Hilbert Theorem 90 are verified. In particular, it is shown in Proposition 5.4, that any $G$-torsor under $\mathcal{A}(F)$ is trivial for any algebraic group $\mathcal{A}$ over $k$. There exist, however, interesting examples of torsors in the case $n<\infty$.

If $n=\infty$ and $\mathcal{A}$ is an irreducible commutative algebraic group over $k$, we show in Corollary 5.2 that $\mathrm{Ext}^1_{\mathcal{S}m_G}(\mathcal{A}(F)/\mathcal{A}(k),\mathbb{Q})=\mathrm{Hom}(\mathcal{A}(k),\mathbb{Q})$, where $\mathcal{S}m_G$ is the category of smooth $G$-modules. If $\mathcal{A}$ is an abelian variety then $\mathcal{A}(F)/\mathcal{A}(k)=\mathfrak{B}^1(\mathcal{A}^\vee)$ (here $\mathcal{A}^\vee:=\mathrm{Pic}^\circ\mathcal{A}$ is the dual abelian variety), so this should correspond to the identity $\mathrm{Ext}^1_{\mathcal{MM}_k}(\mathbb{Q}(0),H_1(\mathcal{A}))=\mathcal{A}(k)_{\mathbb{Q}}$ in the category of mixed motives over $k$. If $\mathcal{A}=\mathbb{G}_m$ then the identity $\mathrm{Ext}^1_{\mathcal{MM}_k}(\mathbb{Q}(0),\mathbb{Q}(1))=k^\times\otimes\mathbb{Q}$ suggests that the non-admissible $G$-module $F^\times/k^\times$ admits a motivic interpretation analogous to $\mathbb{Q}(1)$.

The purpose of §6 is to introduce an abelian category $\mathcal{I}_G$ of "homotopy invariant" representations having some properties of the Chow groups. If $n=\infty$ then it contains all admissible $G$-modules (Proposition 6.4), it is a Serre subcategory in $\mathcal{S}m_G$ (Proposition 6.15), and it is closed under the inner $\mathcal{H}om$ functor on $\mathcal{S}m_G$ (Proposition 6.26). There are no smooth projective representations of $G$, if $n=\infty$ (cf. Remark on p.41). However, it is shown in Corollary 6.11 that $\mathcal{I}_G$ has enough projective objects. Namely, the inclusion functor $\mathcal{I}_G\longrightarrow\mathcal{S}m_G$ admits the left adjoint $\mathcal{S}m_G\xrightarrow{\mathcal{I}}\mathcal{I}_G$, and

to any subextension $L$ of finite type one associates the projective object $C_L := \mathcal{I}\mathbb{Q}[G/G_{F/L}]$ of $\mathcal{I}_G$. For any smooth proper model $X$ of $L/k$ there is a natural surjection $C_L \longrightarrow CH_0(X \times_k F)_{\mathbb{Q}}$. One can expect (Conjecture 6.16) that this is an isomorphism if $n = \infty$. If $\mathrm{tr.deg}(L/k) = 1$ this is Corollary 6.21.

At the end of §6.2, p.50, a functorial decreasing filtration $\mathcal{F}^\bullet$ on objects of $\mathcal{I}_G$ is introduced. It is likely that in the case of $G$-modules of type $CH_0(X_F)_{\mathbb{Q}}$ for smooth proper $X$ over $k$ it is the motivic filtration. This agrees with Corollary 6.24: $C_{k(X)} \cong \mathbb{Q} \oplus \mathrm{Alb}X(F)_{\mathbb{Q}} \oplus \mathcal{F}^2 C_{k(X)}$.

If $n = \infty$ then Conjecture 6.16, the semi-simplicity conjecture and Bloch–Beilinson filtration conjecture would imply (i) Conjecture 1.2, (ii) that any irreducible object of $\mathcal{I}_G$ is admissible, and (iii) that the $G$-modules $gr^N_j W$ are semi-simple for any object $W$ of $\mathcal{I}_G$ (the latter is Conjecture 6.9). Indeed, for some collection of subfields $L \subset F$ of finite type and of transcendence degree $j$ over $k$ there is a surjective morphism $\oplus_L \mathbb{Q}[G/G_{F/L}] \xrightarrow{\xi} gr^N_j W$, which factors through $\oplus_L gr^N_j C_L$, cf. Proposition 6.8. If for a smooth proper model $Y_{[L]}$ of $L/k$ one has $C_L = CH_0(Y_{[L]} \times_k F)_{\mathbb{Q}}$ then $gr^N_j C_L = CH^j(L \otimes_k F)_{\mathbb{Q}}$, so $\xi$ factors through $\oplus_L CH^j(L \otimes_k F)_{\mathbb{Q}}$. One can deduce from the semi-simplicity conjecture and the filtration conjecture (cf. [B] §1.4, or [R] Prop.1.1.1)[1] that $CH^j(L \otimes_k F)_{\mathbb{Q}}$ coincides with $\mathfrak{B}^j(M)$, where $M$ is the maximal primitive $j$-submotive of the motive $(Y_{[L]}, \Delta_{Y_{[L]}})$. Finally, by semi-simplicity, there are projectors $\pi_L$ and an isomorphism

$$\oplus_L \mathfrak{B}^j((Y_{[L]}, \pi_L)) \xrightarrow{\sim} gr^N_j W.$$

This shows (iii), and taking irreducible $W$ (which coincides with $gr^N_j W$ for some $j$) we get also (i) and (ii).

It is also conjectured[2] that the level filtration $N_\bullet$ is strictly compatible with the morphisms in $\mathcal{I}_G$ (cf. Corollary 6.10), so that, in particular, extensions of $G$-modules in $\mathcal{I}_G$ of lower level by irreducible $G$-modules in $\mathcal{I}_G$ of higher level are (canonically) split. Obviously, this is motivated by Hodge theory, and one would like to find a category bigger than $\mathcal{I}_G$ and modify the filtration to keep this property.

However, if we want to consider $G$-modules like $F^\times/k^\times$, the notion of weight should be more subtle. Usually, for a pair $W_1, W_2$ of irreducible objects, $W_1$ is of higher weight if $\mathrm{Ext}^1(W_1, W_2) \neq 0$, so this would give weight$(\mathbb{Q})$ $<$weight$(F^\times/k^\times)$ $<$weight$(\mathcal{A}(F)/\mathcal{A}(k))$, cf. §5.1, for any abelian variety $\mathcal{A}$ over $k$, which is not good if $\mathcal{A}(F)/\mathcal{A}(k)$ corresponds to $H_1(\mathcal{A})$.

If $n = \infty$, the category $\mathcal{I}_G$ carries a tensor structure compatible with the inner $\mathcal{H}om$, but its associativity depends on Conjecture 6.16.

If $n < \infty$ then the category of smooth $G$-modules has sufficiently many projective objects. Namely, any smooth $G$-module is a quotient of a direct

[1] where it is shown that under above assumptions the localization surjection $CH^*(Y_{[L]} \times_k Y_{[L']}) \longrightarrow CH^*(L \otimes_k L')_{\mathbb{Q}}$ kills the numerically trivial cycles.

[2] and deduced from Conjecture 6.9

sum of $\mathbb{Q}[G/U_j]$ for some open compact subgroups $U_j$ of $G$. However, the $G$-modules $\mathbb{Q}[G/U]$ seem to be very complicated. The last section contains two examples of pairs of essentially different open compact subgroups $U_1$ and $U_2$ of $G$ with the same irreducible subquotients of $\mathbb{Q}[G/U_1]$ and $\mathbb{Q}[G/U_2]$. As in both examples the primitive motives of maximal level of models of $F^{U_1}$ and $F^{U_2}$ are trivial, one could expect that collections of irreducible subquotients of $\mathbb{Q}[G/U]$ are of motivic nature.

In Appendix A one shows that the centers of the Hecke algebras of the pairs $(G,U)$ and $(G^\circ,U)$ (see §1.1 for the definition) consist of scalars for any compact subgroup $U$ in $G$. Compared to the analogous question for $p$-adic groups, this is a negative result.

### *1.1 Notations, conventions and terminology*

For a field $F$ and a collection of its subrings $F_0,(F_\alpha)_{\alpha\in I}$ we denote by $G_{\{F,(F_\alpha)_{\alpha\in I}\}/F_0}$ the group of automorphisms of the field $F$ over $F_0$ preserving all $F_\alpha$, and set $G_{F/F_0}:=G_{\{F\}/F_0}$. If $K$ is a subfield of $F$ then $\overline{K}$ denotes its algebraic closure in $F$, $\mathrm{tr.deg}(F/K)$ the transcendence degree of the extension $F/K$ (possibly infinite, but countable), and $U_K$ denotes the group $G_{F/K}$. Throughout the paper $k$ is an algebraically closed field, $F$ its algebraically closed extension with $\mathrm{tr.deg}(F/k)=n\ge 1$ and $G=G_{F/k}$. Everywhere, except the appendix, $k$ is of characteristic zero.

For a totally disconnected topological group $H$ we denote by $H^\circ$ its subgroup generated by the compact subgroups. Obviously, $H^\circ$ is a normal subgroup in $H$, which is open at least if $H$ is locally compact.

In what follows, $\mathbb{Q}$ is the field of rational numbers, and a *module* is always a $\mathbb{Q}$-vector space. For an abelian group $A$ set $A_{\mathbb{Q}}=A\otimes\mathbb{Q}$.

A representation of $H$ in a vector space $W$ over a field is called *smooth*, if stabilizers of all vectors in $W$ are open. A smooth representation $W$ is called *admissible* if fixed vectors of each open subgroup form a finite-dimensional subspace in $W$. A representation of $H$ in $W$ is called *continuous*, if stabilizers of all vectors in $W$ are closed. Any cyclic $G$-module is a quotient of the continuous $G$-module $\mathbb{Q}[G]$, so any $G$-module is a quotient of a continuous $G$-module.

$\mathbb{Q}(\chi)$ is the quotient of the free abelian group generated by the set of compact open subgroups in $G^\circ$ by the relations $[U]=[U:U']\cdot[U']$ for all $U'\subset U$. If $n<\infty$ it is a one-dimensional $\mathbb{Q}$-vector space oriented by $[U]>0$ for any $U$. The group $G$ acts on it by conjugation. $\chi:G\longrightarrow\mathbb{Q}_+^\times$ is the modulus.

$\mathbf{D}_E(H):=\varprojlim_U E[H/U]$ and $\widehat{H}:=\varprojlim_U H/U$, where, for a field $E$ of characteristic zero, the inverse systems are formed with respect to the projections $E[H/V]\stackrel{r_{VU}}{\longrightarrow}E[H/U]$ and $H/V\stackrel{r_{VU}}{\longrightarrow}H/U$ induced by inclusions $V\subset U$ of open subgroups in $H$. $\widehat{H}$ is a semigroup. For any $\nu\in\mathbf{D}_E(H)$, any $\sigma\in H$ and an open subgroup $U$ we set

$$\nu(\sigma U):=\text{coefficient of }[\sigma U]\text{ of the image of }\nu\text{ in }E[H/U].$$

The *support* of $\nu$ is the minimal closed subset $S$ in $\widehat{H}$ such that $\nu(\sigma U)=0$ if $\sigma U\bigcap S=\emptyset$. Define a pairing $\mathbf{D}_E(H)\times W\longrightarrow W$ for each smooth $E$-representation $W$ of $H$ by $(\nu,w)\longmapsto\sum_{\sigma\in H/V}\nu(\sigma V)\cdot\sigma w$, where $V$ is an arbitrary open subgroup in the stabilizer of $w$. When $W=E[H/U]$ this pairing is compatible with the projections $r_{VU}$, so we get a pairing $\mathbf{D}_E(H)\times\varprojlim_U E[H/U]\longrightarrow\varprojlim_U E[H/U]=\mathbf{D}_E(H)$, and thus an associative multiplication $\mathbf{D}_E(H)\times\mathbf{D}_E(H)\xrightarrow{*}\mathbf{D}_E(H)$ extending the convolution of compactly supported measures. Set $\mathbf{D}_E=\mathbf{D}_E(G)$ and $\mathbf{D}_E^{\circ}=\mathbf{D}_E(G^{\circ})$.

If $U$ is a compact subgroup in $H$ the *Hecke algebra* of the pair $(H,U)$ is the subalgebra $\mathcal{H}_E(H,U):=h_U*\mathbf{D}_E(H)*h_U$ in $\mathbf{D}_E(H)$ of $U$-bi-invariant measures. Here $h_U$ is the Haar measure on $U$ defined by the system $(h_U)_V=[U:U\bigcap V]^{-1}\sum_{\sigma\in U/U\bigcap V}[\sigma V]\in\mathbb{Q}[H/V]$ for all open subgroups $V\subset H$. $h_U$ is the identity in $\mathcal{H}_E(H,U)$ and $h_Uh_{U'}=h_U$ for a closed subgroup $U'\subseteq U$. Set $\mathcal{H}_E(U)=\mathcal{H}_E(G,U)$, $\mathcal{H}(U)=\mathcal{H}_{\mathbb{Q}}(G,U)$ and $\mathcal{H}_E^{\circ}(U)=\mathcal{H}_E(G^{\circ},U)$.

For any variety $X$ over $k$ and any field extension $E/k$ we set $X_E:=X\times_k E$, and denote by $\widetilde{X}$ one of its desingularizations. For an extension $L/k$ of finite type, $Y_{[L]}=Y_{U_L}$ denotes a smooth proper model of $L/k$.

$\mathbb{P}_K^M$ denotes the $M$-dimensional projective space over a field $K$.

For a commutative group scheme $\mathcal{A}$ over $k$ we set $W_{\mathcal{A}}=\mathcal{A}(F)/\mathcal{A}(k)$.

$\mathcal{S}m_H(E)$ is the category of smooth $E$-representations of $H$. $\mathcal{I}_G(E)$ is the full subcategory in $\mathcal{S}m_G(E)$ consisting of those representations $W$ of $G$ for which $W^{G_{F/L}}=W^{G_{F/L'}}$ for any extension $L$ of $k$ in $F$ and any purely transcendental extension $L'$ of $L$ in $F$. When discussing $\mathcal{I}_G(E)$, the principal case will be $n=\infty$. We set $\mathcal{S}m_H=\mathcal{S}m_H(\mathbb{Q})$ and $\mathcal{I}_G=\mathcal{I}_G(\mathbb{Q})$. The *level* filtration $N_\bullet$ is defined in the beginning of §6.1.

## 2 Preliminaries on closed subgroups in $G$

The topology on $G$ described in Introduction has been studied in [Jac], p.151, Exercise 5, [ПШ-Ш], [Sh] Ch.6, §6.3, and [I] Ch.2, Part 1, Section 1. It is shown that $G$ is Hausdorff, locally compact if $n<\infty$, and totally disconnected; the subgroups $G_{\{F,(F_\alpha)_{\alpha\in I}\}/k}$ are closed in $G$, there is an injective morphism of unitary semigroups

$$\{\text{subfields in } F \text{ over } k\}\longrightarrow\{\text{closed subgroups in } G\}$$

given by $K\longmapsto\mathrm{Aut}(F/K)$, its image is stable under passages to sub-/supgroups with compact quotients, and it induces bijections

- $\{\text{subfields } K\subset F \text{ over } k \text{ with } F=\overline{K}\}\leftrightarrow\{\text{compact subgroups of } G\}$;
- $\left\{\begin{array}{c}\text{subfields } K \text{ of } F \text{ of finite type}\\ \text{over } k \text{ with } F=\overline{K}\end{array}\right\}\leftrightarrow\left\{\begin{array}{c}\text{compact open}\\ \text{subgroups of } G\end{array}\right\}$.

  The inverse correspondences are given by $G\supset H\longmapsto F^H$ (the subfield in $F$ fixed by $H$).

The first is [ПШ-Ш], §3, Lemma 1, or [Sh] Prop.6.11; the second is immediate from loc.cit., or [Sh] Prop.6.12.

**Lemma 2.1.** *If $L \subseteq F$ containing $k$ is the intersection of a collection $\{L_\alpha\}_\alpha$ of its algebraic extensions then the subgroup in $G$ generated by all $U_{L_\alpha}$ is dense in $U_L$.*

*Proof.* Clearly, $U_L$ contains the subgroup in $G$ generated by all $U_{L_\alpha}$, and $G_{F/\overline{L}}$ is a normal subgroup in each of $U_{L_\alpha}$ and in $U_L$. Set $\overline{U}_{L_\alpha} = U_{L_\alpha}/G_{F/\overline{L}}$, and similarly $\overline{U}_L = U_L/G_{F/\overline{L}}$. These are compact subgroups in $G_{\overline{L}/k}$. Then, under the above Galois correspondence, the closure of the subgroup in $G_{\overline{L}/k}$ generated by all $\overline{U}_{L_\alpha}$ corresponds to the subfield $\overline{L}^{\langle \overline{U}_{L_\alpha} \mid \alpha\rangle} = \bigcap_\alpha L_\alpha = L$, i.e., to the same subfield as $\overline{U}_L$, which means that $\langle \overline{U}_{L_\alpha} \mid \alpha\rangle$ is dense in $\overline{U}_L$, and therefore, $\langle U_{L_\alpha} \mid \alpha\rangle$ is dense in $U_L$. □

**Lemma 2.2.** *Let $L \subset F$ be such an extension of $k$ that any subextension of transcendence degree $\leq 2$ is of finite type over $k$. Then the common normalizer in $G$ of all normal closed subgroups of index $\leq 3$ in $U_L$ coincides with $U_L$.*

*Proof.* Any element $\tau$ in the common normalizer in $G$ of all closed subgroups of index $\leq 2$ in $U_L$ satisfies $\tau(L(f^{1/2})) = L(f^{1/2})$ for all $f \in L^\times$. If $\tau \notin U_L$ then there is an element $x \in L^\times$ such that $\tau x/x \neq 1$. Then $\tau x/x = y^2$ for some $y \in F^\times - \{\pm 1\}$. Set $f = x+\lambda$ for a variable $\lambda \in k$. By Kummer theory, $\tau f/f \in L^{\times 2}$, and therefore, $L$ contains $L_0 := k\left(y\frac{(x+\lambda y^{-2})^{1/2}}{(x+\lambda)^{1/2}} \mid \lambda \in k\right) \subset \overline{k(x,y)}$.

As $\mathrm{tr.deg}(\overline{k(x,y)}/k) \leq 2$, by our assumption on $L$, the subfield $L_0$ of $L$ should be finitely generated over $k$. But this is possible only if $y^2 = 1$, i.e., if $\tau \in U_L$. (To see this, one can choose a smooth model of the extension $L_0(x)/k(x,y)$ over $k$ and look at its branch locus.) □

**Corollary 2.3.** *For any $\xi \in G - \{1\}$ and any open subgroup $U \subseteq G$ there exists an element $\sigma \in U$ such that $[\sigma, \xi] \in U - \{1\}$.*

*Proof.* Let $L$ be such a subfield of $F$ finitely generated over $k$ that $U_L \subseteq U$.

Set $L' = L\xi(L)$. Then $\xi^{-1}(\sigma\xi\sigma^{-1})|_L = id$ for any $\sigma \in U_{L'}$. By Lemma 2.2, the centralizer in $G$ of $U_{L'}$ is trivial. (Let $\{L_\alpha\}$ be the set of all finitely generated extensions of $L'$. Then $\bigcup_\alpha L_\alpha = F$. Any element $\tau \in G$ centralizing $U_{L'}$ normalizes all subgroups in all $U_{L_\alpha}$, and thus, by Lemma 2.2, $\tau \in \bigcap_\alpha U_{L_\alpha} = \{1\}$.) In particular, as $\xi$ does not centralize $U_{L'}$, there is $\sigma \in U_{L'}$ with $\xi^{-1}\sigma\xi\sigma^{-1} \in U_L - \{1\}$. □

**Lemma 2.4.** *Let $H \neq \{1\}$ be a normal closed subgroup in $G^\circ$ such that $H \bigcap U \neq \{1\}$ for a compact subgroup $U$. Then $H$ contains $U_{k'(x)}$ for any algebraically closed extension $k'$ of $k$ in $F$ such that $k' = \overline{k' \bigcap F^U}$ and* $\mathrm{tr.deg}(F/k') = 1$*, and for some $x \in F - k'$.*

*Proof.* Let $\sigma \in H \bigcap U - \{1\}$ and $k'$ be the algebraic closure in $F$ of any subfield in $F^{\langle\sigma\rangle}$ with $\mathrm{tr.deg}(F/k') = 1$. As the extension $F/F^{\langle\sigma\rangle}$ is abelian

there is an element $x \in F - k'$ and an integer $N \geq 2$ such that $\sigma x \neq x$ and $\sigma x^N = x^N$. Then one has $\sigma(k'(x)) = k'(x)$.

Let $L$ be a finite Galois extension of $k'(x)$. Its smooth proper model over $k'$ is unramified outside a finite set $S$ of points on a smooth proper model of $k'(x)$ over $k'$. Then there is an element $\overline{\alpha}_L \in \mathrm{Aut}(k'(x)/k')$ such that the set $\overline{\alpha}_L^{-1}\sigma\overline{\alpha}_L(S)$ does not intersect $S$, and therefore, for an extension $\alpha_L \in G^{\circ}_{F/k'}$ of $\overline{\alpha}_L$ to $F$, a smooth proper model over $k'$ of the field $L \bigcap \alpha_L^{-1}\sigma\alpha_L(L)$ is unramified over the model of $k'(x)$, so $L \bigcap \alpha_L^{-1}\sigma\alpha_L(L) = k'(x)$.

Let $\beta \in G^{\circ}_{F/k'}$ be given on $k'(x)$ by $\overline{\alpha}_L$, and somehow extended to the field $\alpha_L^{-1}\sigma\alpha_L(L)$. Then $\beta^{-1} \circ \sigma^{-1} \circ \beta \circ \alpha_L^{-1} \circ \sigma \circ \alpha_L$ is the identity on $k'(x)$, and therefore, induces an automorphism of $L$. Since $L$ and $\alpha_L^{-1}\sigma\alpha_L(L)$ are Galois extensions of $L \bigcap \alpha_L^{-1}\sigma\alpha_L(L) = k'(x)$, for any given automorphism $\tau$ of $L$ over $k'(x)$ there is an extension of $\beta$ to $F$ such that for its restriction to $L$ the composition $\beta^{-1} \circ \sigma^{-1} \circ \beta \circ \alpha_L^{-1} \circ \sigma \circ \alpha_L$ coincides with $\tau$. This means that the natural projection $H \bigcap U_{k'(x)} \longrightarrow \mathrm{Gal}(L/k'(x))$ is surjective for any Galois extension $L$ of $k'(x)$, i.e., that $H \bigcap U_{k'(x)}$ is dense in $U_{k'(x)}$. As $H \bigcap U_{k'(x)}$ is closed, we have $H \supseteq U_{k'(x)}$. $\square$

**Lemma 2.5.** *If $n = 1$ then there are no proper normal open subgroups in $G^{\circ}$.*

*Proof.* Let $H$ be a normal open subgroup in $G^{\circ}$. Then for some subfield $L \subset F$ finitely generated over $k$ one has $U_L \subseteq H$. For any purely transcendental extension $L' \subset F$ of $k$ with $\overline{L'} = F$ one also has $U_{LL'} \subseteq H$, as well as $H \supseteq \langle \sigma U_{LL'}\sigma^{-1} \mid \sigma \in N_{G^{\circ}}U_{L'}\rangle$.

A smooth proper model over $k$ of the extension $LL'/L'$ is ramified only over a divisor on the model $\mathbb{P}^1_k$ of $L'$ over $k$, but the group $N_{G^{\circ}}U_{L'}/U_{L'} \cong \mathrm{PGL}_2 k$ does not preserve this divisor, so the intersection $\bigcap_{\sigma \in N_{G^{\circ}}U_{L'}} \sigma(LL')$ is unramified over $L'$, i.e., the field $\bigcap_{\sigma \in N_{G^{\circ}}U_{L'}} \sigma(LL')$ coincides with $L'$. By Lemma 2.1, this shows that $H \supseteq U_{L'}$ for any purely transcendental extension $L' \subset F$ of $k$ with $\overline{L'} = F$, and therefore, $H$ contains all compact subgroups in $G$, so finally, $H \supseteq G^{\circ}$. $\square$

**Lemma 2.6.** *Let $H$ be a closed subgroup of $G^{\circ}$. Assume that for any algebraically closed extension $k'$ of $k$ in $F$ with* $\mathrm{tr.deg}(F/k') = 1$ *and any $x \in F - k'$ the subgroup $H$ contains $U_{k'(x)}$. Then $H = G^{\circ}$.*

*Proof.* Any element $\sigma$ of a compact subgroup in $G^{\circ}$ can be presented as the limit of a compatible collection $(\sigma_L)$ of embeddings $\sigma_L$ of finitely generated extensions $L$ of $k$ in $F$ into $F$. Replacing $L$ with the compositum of the images of $L$ under powers of $\sigma$, we may suppose that $L$ is $\sigma$-invariant.

For a finitely generated $k$-algebra $R$ with the fraction field $L^{\langle\sigma\rangle}$, let $X$ be the affine variety over $k$ with the coordinate ring $R[t, \sigma t, \sigma^2 t, \dots] \subset L$ for some $t \in L$ such that $L = L^{\langle\sigma\rangle}(t)$. Let $Y := \mathbf{Spec}(R) \subseteq \mathbb{A}^N_k$.

By induction on $\dim X$ we show that there is a fibration of a Zariski open $\langle\sigma\rangle$-invariant subset of $X$ by $\langle\sigma\rangle$-invariant irreducible curves. Namely, if $\dim X = 1$ there is nothing to prove. If $\dim X > 1$ then, by Théorème 6.3

4) of [Jou],[3] pull-backs of sufficiently general hyperplanes in $\mathbb{A}_k^N$ under the composition $X \longrightarrow Y \hookrightarrow \mathbb{A}_k$ are irreducible, and thus, there is a fibration of a Zariski open $\langle\sigma\rangle$-invariant subset $U$ of $X$ by $\langle\sigma\rangle$-invariant irreducible divisors with the base $S$. By induction assumption, there is a fibration of a Zariski open $\langle\sigma\rangle$-invariant subset of $U \times_S \overline{k(S)}$ by $\langle\sigma\rangle$-invariant irreducible curves over $\overline{k(S)}$, i.e., there is a fibration of a Zariski open $\langle\sigma\rangle$-invariant subset $U'$ of $X$ by $\langle\sigma\rangle$-invariant irreducible curves over $k$.

Then the function field $k''$ of the base is a subfield in $L^{\langle\sigma\rangle}$ algebraically closed in $L$ and containing $k$. Let $x \in L^{\langle\sigma\rangle} - k''$, and let $k'$ be a maximal algebraically closed subfield in $F$ containing $k''$ but not $x$. Then $\sigma|_L$ coincides with restriction to $L$ of an element of $U_{k'(x)}$. This shows that $\sigma$ belongs to the closure of the union of $U_{k'(x)}$ over all $k'$ and all $x \in F - k'$. □

**Lemma 2.7.** *If $\xi$ is a non-trivial element of $G$ and $2m \le n$ then there exist elements $w_1, \dots, w_m \in F$ such that $w_1, \dots, w_m, \xi w_1, \dots, \xi w_m$ are algebraically independent over $k$.*

*Proof.* We proceed by induction on $m$, the case $m = 0$ being trivial. We wish to find $w_m \in F$ such that $w$ and $\xi w_m$ are algebraically independent over $k'$ generated over $k$ by $w_1, \dots, w_{m-1}, \xi w_1, \dots, \xi w_{m-1}$. Suppose that there is no such $w_m$. Then for any $u \in F - \overline{k'}$ and any $v \in F - \overline{k'(\xi u)}$ one has the following vanishings in $\Omega^2_{F/k'}$: $du \wedge d\xi u = dv \wedge d\xi v = 0$, $d(u+v) \wedge d\xi(u+v) = 0$, and $d(u + v^2) \wedge d\xi(u + v^2) = 0$. Applying the first two to the third, we get $2(v - \xi v) dv \wedge d\xi u = 0$, which means that $\xi v = v$ for any $v \in F - \overline{k'(\xi u)}$, i.e., $\xi = 1$. This contradiction shows that there exists desired $w_m \in F$. □

**Lemma 2.8.** *Let $L$ be a subfield of $F$ with* $\mathrm{tr.deg}(F/L) = \infty$. *Then $G^{\circ}_{F/L}$ is dense in $G_{F/L}$.*

*Proof.* Fix a transcendence basis $x_1, x_2, x_3, \dots$ of $F$ over $L$. We wish to show that for each $\sigma \in G_{F/L}$, any integer $m \ge 1$ and any $y_1, \dots, y_m \in F$ there is a triplet $(\tau_1, \tau_2, \tau_3)$ of elements of some compact subgroups in $G_{F/L}$ such that $\tau_3\tau_2\tau_1\sigma y_s = y_s$ for all $1 \le s \le m$. As $y_1, \dots, y_m$ are algebraic over $k_0 := L(x_1, \dots, x_M)$ for some integer $M \ge m$, it is enough to show that for each $\sigma \in G_{F/L}$ and any integer $M \ge 1$ there is a pair $(\tau_1, \tau_2)$ of elements of some compact subgroups in $G_{F/L}$ such that $\tau_2\tau_1\sigma x_s = x_s$ for all $1 \le s \le M$.

Let $k_1 := \overline{k_0\sigma(k_0)}$, $z_j \in F - k_j$ and $k_{j+1} := \overline{k_j(z_j)}$ for all $1 \le j \le M$. Let $\tau_1\sigma x_j = z_j$ and $\tau_1 z_j = \sigma x_j$ for all $1 \le j \le M$ and $\tau_1$ is somehow extended to an element of a compact subgroup in $G_{F/L}$. Let $\tau_2 x_j = z_j$ and $\tau_2 z_j = x_j$ for all $1 \le j \le M$ and $\tau_2$ is somehow extended to an element of a compact subgroup in $G_{F/L}$. Then $(\tau_2\tau_1)\sigma x_j = x_j$. □

**Theorem 2.9.** *If $n < \infty$ then any non-trivial subgroup in $G$ normalized by $G^{\circ}$ is dense in $G^{\circ}$. If $n = \infty$ then any non-trivial normal subgroup in $G$ is dense.*

[3] For any irreducible scheme $X$ of finite type over $k$ and a morphism $X \xrightarrow{f} \mathbb{A}_k$ to an affine space with $\dim \overline{f(X)} \ge 2$ the preimages of almost all hyperplanes in $\mathbb{A}_k$ are irreducible.

*Proof.* Let $H$ be a non-trivial closed subgroup in $G$ normalized by $G^\circ$. By Corollary 2.3, $H$ intersects non-trivially any open compact subgroup in $G^\circ$ if $n<\infty$. Then, by Lemma 2.4, the group $H\bigcap G^\circ_{F/k'}$ is an open normal subgroup in $G^\circ_{F/k'}$ for any algebraically closed extension $k'$ of $k$ in $F$ with $\mathrm{tr.deg}(F/k')=1$, and thus, by Lemma 2.5, $H\bigcap G^\circ_{F/k'}=G^\circ_{F/k'}$. Then Lemma 2.6 implies that $H\supseteq G^\circ$.

In the case $n=\infty$ and $H$ is normal, we fix a pair of transcendence basis $x_1,x_2,x_3,\dots$ and $y_1,y_2,y_3,\dots$ of $F$ over $k$ and show that for each $m$ there is an element $\sigma\in H$ such that $\sigma x_j=y_j$ for all $1\le j\le m$. Fix a non-trivial element $\xi\in H$. Choose some elements $z_1,\dots,z_m$ algebraically independent over $k(x_1,\dots,x_m,y_1,\dots,y_m)$. By Lemma 2.7, there exist $w_1,\dots,w_m\in F$ such that $w_1,\dots,w_m,\xi w_1,\dots,\xi w_m$ are algebraically independent over $k$. Then there exist elements $\tau_1,\tau_2\in G$ such that $\tau_1x_j=w_j$, $\tau_1z_j=\xi w_j$ and $\tau_2y_j=w_j$, $\tau_2z_j=\xi w_j$ for all $1\le j\le m$. Then $\sigma:=\tau_2^{-1}\circ\xi^{-1}\circ\tau_2\circ\tau_1^{-1}\circ\xi\circ\tau_1$ sends $x_j$ to $y_j$ for all $1\le j\le m$.

This implies that there is a compact subgroup $U$ intersecting $H$ non-trivially, so by Lemma 2.4, $H$ contains $U_{k'(x)}$ for some algebraically closed extension $k'$ of $k$ in $F$ with $\mathrm{tr.deg}(F/k')=1$, and for some $x\in F-k'$. By Lemma 2.5, this implies that $H$ contains $G^\circ_{F/k'}$.

The algebraically closed extensions $k'$ of $k$ in $F$ with $\mathrm{tr.deg}(F/k')=1$ form a single $G$-orbit, so one can apply Lemma 2.6, which gives $H\supseteq G^\circ$. Finally, it follows from Lemma 2.8 that $H=G$. □

**Corollary 2.10.** *Any smooth representation of $G^\circ$ of finite degree is trivial.*

*(Proof.* This is clear, since there are no proper open normal subgroups in $G^\circ$.*)* □

**Corollary 2.11.** *For any subgroup $H$ of $G$ containing $G^\circ$ and any continuous homomorphism $\pi$ from $H$ either $\pi$ is injective, or the restriction of $\pi$ to $G^\circ$ is trivial.*

*Proof.* If $\pi$ is not injective then, by Corollary 2.3, its kernel has a non-trivial intersection with $G^\circ$. Then, by Theorem 2.9, the kernel of $\pi$ contains $G^\circ$. □

**Lemma 2.12.** *Let $d\ge 0$ be an integer, $F_1$ and $F_2$ be algebraically closed subfields of $F$ such that $F_1\bigcap F_2=k$ and $\mathrm{tr.deg}(F/F_2)\ge d$. Then for any subfield $L$ in $F$ with $\mathrm{tr.deg}(L/k)=d$ there is $\xi\in H:=\langle G_{F/F_1},G_{F/F_2}\rangle$ such that $\mathrm{tr.deg}(\xi(L)F_2/F_2)=d$.*

*Proof.* We proceed by induction on $d$, the case $d=0$ being trivial. If $d>0$ fix a subfield $L_1\subset L$ with $\mathrm{tr.deg}(L_1/k)=d-1$ and some $t\in L$ transcendental over $L_1$.

Replacing $F_1$ with the algebraic closure in $F$ of the subfield generated over $F_1$ by a transcendence basis of $F$ over $F_1F_2$ (thus, making $H$ smaller), we may assume that $F$ is algebraic over $F_1F_2$. In particular, there exists a subfield $K\subset F_1$ over $k$ with $\mathrm{tr.deg}(K/k)=d$ and $\mathrm{tr.deg}(KF_2/F_2)=d$.

By the induction assumption, there exists an element $\tau \in H$ such that $\mathrm{tr.deg}(\tau(L_1)F_2/F_2) = d-1$, i.e., we may suppose that $\mathrm{tr.deg}(L_1F_2/F_2) = d-1$. Moreover, we may suppose that $L_1 = k(t_1, \dots, t_{d-1})$ is purely transcendental over $k$ and $L = L_1(t)$.

The subgroup $G_{F/F_2}$ acts transitively on the set of purely transcendental extensions of $F_2$ of a given transcendence degree, so for any collection $x_1, \dots, x_{d-1}$ of elements of $F_1$ algebraically independent over $F_2$ there is $\sigma \in G_{F/F_2}$ such that $\sigma t_j = x_j$.

If $t \notin \overline{L_1F_2}$ then induction is completed, so we assume that $t$ is algebraic over $L_1F_2$, i.e., there is an irreducible polynomial $P \in F_2[X_1, \dots, X_d] - k[X_1, \dots, X_d]$ with $P(t_1, \dots, t_d) = 0$, where $t_d := t$.

Consider the irreducible hypersurface

$$W = \{(y_1, \dots, y_d) \mid P(y_1, \dots, y_d) = 0\} \hookrightarrow \mathbb{A}^d_{F_2} \longrightarrow \mathbb{A}^d_k$$

and the projection $W \xrightarrow{\pi} \mathbb{A}^{d-1}_{F_2}$ to the first $d-1$ coordinates. Suppose that for any $\sigma$ as above one has $\sigma t \in F_1$. Then for any generic point $(x_1, \dots, x_{d-1}) \in \mathbb{A}^{d-1}_F$ as above the points of the fiber of the projection $\pi$ over $(x_1, \dots, x_{d-1})$ are defined over $F_1$. This means that $W$ is defined over $F_1$, and therefore, over $F_1 \bigcap F_2 = k$, contradicting $\mathrm{tr.deg}(L/k) = d$.

Therefore, there is $\sigma \in G_{F/F_2}$ such that $\sigma(L_1) \subset F_1$ and $\sigma t \notin F_1$, so in the rest of the proof we assume that $L_1 \subset F_1$ and $t \notin F_1$.

The $G_{F/F_1}$-orbit of $t$ coincides with $F - F_1$. As the intersection

$$(F - F_1) \bigcap \left(F - \overline{L_1F_2}\right) = F - \left(F_1 \bigcup \overline{L_1F_2}\right)$$

is non-empty, there is an element $\xi$ in $G_{F/F_1}$ such that $\xi t \in F - \overline{L_1F_2}$, so finally, $\mathrm{tr.deg}(\xi(L)F_2/F_2) = d$. $\square$

**Corollary 2.13.** *In notations of Lemma 2.12, for any $\sigma \in G$ there is $\tau \in H$ such that $\sigma|_L = \tau|_L$.*

*Proof.* According to Lemma 2.12, there exist elements $\xi, \xi' \in H$ such that

$$\mathrm{tr.deg}(\xi'(L)F_2/F_2) = \mathrm{tr.deg}(\xi\sigma(L)F_2/F_2) = \mathrm{tr.deg}(L/k).$$

Evidently, there is $\lambda \in G_{F/F_2} \subset H$ inducing an isomorphism $\xi'(L) \xrightarrow{\sim} \xi\sigma(L)$ such that $\sigma|_L = \xi^{-1} \circ \lambda \circ \xi'|_L$. $\square$

**Proposition 2.14.** *Let $L_1$ and $L_2$be subextensions of $k$ in $F$ such that $\overline{L_1} \bigcap \overline{L_2}$ is algebraic over $L_1 \bigcap L_2$ and $\mathrm{tr.deg}(F/L_2) = \infty$. Then the subgroup in $G$ generated by $G_{F/L_1}$ and $G_{F/L_2}$ is dense in $G_{F/L_1 \bigcap L_2}$.*

*Proof.* The inclusion $\langle G_{F/L_1}, G_{F/L_2} \rangle \subseteq G_{F/L_1 \bigcap L_2}$ is evident. In Corollary 2.13 we may replace $k$ with $\overline{L_1} \bigcap \overline{L_2}$ to get that the subgroup in $G$ generated by $G_{F/\overline{L_1}}$ and $G_{F/\overline{L_2}}$ is dense in $G_{F/\overline{L_1} \bigcap \overline{L_2}}$.

It remains to show that the compact group $\overline{\langle G_{F/L_1}, G_{F/L_2} \rangle}/G_{F/\overline{L_1} \bigcap \overline{L_2}}$ coincides with $\mathrm{Gal}(\overline{L_1} \bigcap \overline{L_2}/L_1 \bigcap L_2)$. But this is the Galois theory:

$$\left(\overline{L_1} \bigcap \overline{L_2}\right)^{\langle G_{F/L_1}, G_{F/L_2} \rangle} = L_1 \bigcap L_2. \qquad \square$$

**Lemma 2.15.** *If $n<\infty$ and $F\neq\overline{k'}$, for a subfield $k'$ over $k$ in $F$, then $G=G_{F/k'}\cdot G^{\circ}$.*

*Proof.* We may suppose that $k'$ is algebraically closed and maximal in $F-\{x\}$ for some $x\in F-k$. Fix a transcendence basis $x_1,\dots,x_{n-1}$ of $k'$ over $k$. We have to show that for each $\sigma\in G$ there is such an element $\tau\in G^{\circ}$ that $\sigma\tau x_s=x_s$ for all $1\le s<n$.

Set $x_n=x$. Let $0\le j<n$ be the maximal integer with the property that there is an element $\tau\in G^{\circ}$ such that $\sigma\tau\in G_{F/k''}$, where $k''=\overline{k(x_1,\dots,x_j)}$. For such $\tau$ set $\sigma\tau=\sigma'$. Suppose that $0\le j<n-1$. Then, by Lemma 2.7, there exists $w\in F-k''$ such that $w$ and $\sigma'w$ are algebraically independent over $k''$. There are integers $s,s'>j$ such that $w$ is transcendental over $k''(x_s)$ and $\sigma'w$ is transcendental over $k''(x_{s'})$.

Then there are elements $\tau_1,\tau_2,\tau_3,\tau_4$ of some compact subgroups of $G_{F/k''}$ such that $\tau_1x_{j+1}=x_s$, $\tau_2x_s=w$, $\tau_3(\sigma'w)=x_{s'}$, $\tau_4x_{s'}=x_{j+1}$. Then the automorphism $\tau_4\tau_3\sigma'\tau_2\tau_1=\sigma\tau_0$, where $\tau_0\in G^{\circ}$, acts trivially on $k''(x_{j+1})$, contradicting our assumptions. This means that $j=n-1$. $\square$

**Lemma 2.16.** *Let $L$ be a subfield of $F$ and $S$ be a set of elements of $F$ algebraically independent over $L$. Then the group $H$ generated by $G_{F/L(S-\{x\})}$ for all $x\in S$ is dense in $G_{F/L}$.*

*Proof.* If $\overline{H}$ contains the subgroups $G_{F/L(S')}$ for all subsets $S'\subset S$ with finite complement then it contains the closure of $\bigcup_{S'}G_{F/L(S')}$ coincident with $G_{F/L(\bigcap_{S'}S')}=G_{F/L}$. So it suffices to treat the case $S=\{x,y\}$.

For any $z\in F-\overline{L}$ there is an element $\sigma\in G_{F/L(x)}G_{F/L(y)}$ such that $\sigma x=z$, so $H$ contains $\sigma G_{F/L(x)}\sigma^{-1}=G_{F/L(z)}$, and thus, $H$ is a normal subgroup in $U_L$. By Theorem 2.9, $\overline{H}$ contains $G_{F/\overline{L}}$ if $\mathrm{tr.deg}(F/L)=\infty$, and $\overline{H}$ contains $G^{\circ}_{F/\overline{L}}$ otherwise. By Lemma 2.15, in the latter case the projection $G_{F/\overline{L}(x)}\longrightarrow G_{F/\overline{L}}/G^{\circ}_{F/\overline{L}}$ is surjective, and thus, $\overline{\langle G_{F/\overline{L}(x)},G_{F/\overline{L}(y)}\rangle}$ surjects onto $G_{F/\overline{L}}/G^{\circ}_{F/\overline{L}}$, so $\overline{\langle G_{F/\overline{L}(x)},G_{F/\overline{L}(y)}\rangle}=G_{F/\overline{L}}$.

As the subgroup $G_{F/L(x)}$ of $H$ surjects onto $G_{F/L}/G_{F/\overline{L}}\cong\mathrm{Aut}(\overline{L}/L)$, and $H$ is an extension of a subgroup in $G_{F/L}/G_{F/\overline{L}}$ by $G_{F/\overline{L}}\subseteq H$, we get $\overline{H}=G_{F/L}$. $\square$

## 3 Some geometric representations

Now we are going to construct a supply of semi-simple admissible representations of $G$. Recall ([BZ]), that for each smooth $E$-representation $W$ of $G$ and each compact subgroup $U$ of $G$ the Hecke algebra $\mathcal{H}_E(U)$ acts on the space $W^U$, since $W^U=h_U(W)$.

Note that the definition in §1.1 on p.7 is equivalent to the usual definition of $G$ the Hecke algebra when $U$ is open.

**Proposition 3.1.** *1. Let $1 \le n \le \infty$. Then a smooth $E$-representation $W$ of $G$ is irreducible if and only if for each compact subgroup $U$ with $F^U$ purely transcendental over an extension of $k$ of finite type the $\mathcal{H}_E(U)$-module $W^U$ is irreducible.*

*2. Let $W_j$ for $j = 1, 2$ be smooth irreducible $E$-representations of $G$ and $W_1^U \neq 0$ for some compact subgroup $U$. Then $W_1$ is equivalent to $W_2$ if and only if $W_1^U$ is equivalent to $W_2^U$.*

*3. For each open compact $U \subset G$ and each irreducible $E$-representation $\tau$ of the algebra $\mathcal{H}_E(U)$ there is a smooth irreducible representation $W$ of $G$ with $\tau \cong W^U$.*

*Proof.* If $n < \infty$ then this is Proposition 2.10 of [BZ]. In the case $n = \infty$ the proof is modified as follows.

1. Suppose that $W$ is irreducible and $W^U \neq 0$. If $V$ is a non-zero $\mathcal{H}_E(U)$-submodule in $W^U$ then the natural morphism $\mathbf{D}_E h_U \otimes_{\mathcal{H}_E(U)} V \longrightarrow W$ is non-zero, and thus, surjective. Then $V = h_U \mathbf{D}_E h_U \otimes_{\mathcal{H}_E(U)} V$ surjects onto $W^U$, i.e., $V = W^U$.
   Conversely, let $W_1 \subset W$ be a non-zero proper subrepresentation. Fix an element $e \in W - W_1$. There is a compact subgroup $U \subset \mathrm{Stab}_e$ with $F^U$ purely transcendental over an extension of $k$ of finite type. Then $e \in W_1^U \neq W^U$, giving contradiction.
2. "Only if" part is evident, so suppose that the $\mathcal{H}_E(U)$-modules $W_1^U$ and $W_2^U$ are isomorphic. Then one has $\mathbf{D}_E h_U \otimes_{\mathcal{H}_E(U)} W_1^U \cong \mathbf{D}_E h_U \otimes_{\mathcal{H}_E(U)} W_2^U$. Set $W_U := \{w \in W \mid h_U \mathbf{D}_E w = 0\}$ and ${}_UW := W/W_U$.
   Then ${}_U(\mathbf{D}_E h_U \otimes_{\mathcal{H}_E(U)} W_j^U)$ is irreducible, since otherwise the inclusion $0 \neq W_3 \subset {}_U(\mathbf{D}_E h_U \otimes_{\mathcal{H}_E(U)} W_j^U)$ would imply $W_3^U = W_j^U$, and therefore, that $W_3 = {}_U(\mathbf{D}_E h_U \otimes_{\mathcal{H}_E(U)} W_j^U)$. This gives that
   $$W_1 = {}_U(\mathbf{D}_E h_U \otimes_{\mathcal{H}_E(U)} W_1^U) \cong {}_U(\mathbf{D}_E h_U \otimes_{\mathcal{H}_E(U)} W_1^U) = W_2. \quad \square$$

**Lemma 3.2.** *Let $W$ be a smooth $E$-representation of $G$. Suppose that for each compact subgroup $U \subset G$ with $F^U$ purely transcendental over an extension of $k$ of finite type the $\mathcal{H}_E(U)$-submodule $W^U$ is semi-simple. Then $W$ is semi-simple.*

*Proof.* Let $W_0$ be the sum of all irreducible $E$-subrepresentations of $G$ in $W$. Suppose $W \neq W_0$ and $e \in W - W_0$. Let $V$ be the subrepresentation of $G$ in $W$ generated by $e$, and $V_0 = V \bigcap W_0$. Let $V_2$ be a maximal subrepresentation of $G$ in $V - \{e\}$ containing $V_0$. Then the representation $V_1 = V/V_2$ of $G$ is irreducible. Our goal is to embed $V_1$ into $W$. Let $U$ be a compact subgroup in $G$ such that $F^U$ is purely transcendental over an extension of $k$ of finite type and $V_1^U \neq 0$. By Proposition 3.1(1), the $\mathcal{H}_E(U)$-module $V_1^U$ is irreducible. As the $\mathcal{H}_E(U)$-module $V^U$ is semi-simple and $V^U/V_2^U$ is isomorphic to $V_1^U$, there is an $\mathcal{H}_E(U)$-submodule $N_U$ in $V^U$ isomorphic to $V_1^U$ and intersecting $V_2$ trivially.

For each open subgroup $U'$ of $U$ the $\mathcal{H}_E(U')$-submodule $N'_{U'}$ of $U'$-invariants of the subrepresentation of $G$ in $V$ generated by $N_U$ splits into

a direct sum $\bigoplus_j A_j^{m_j}$ of irreducible $\mathcal{H}_E(U')$-modules, where $m_j \le \infty$ and $A_i \ncong A_j$ if $i \ne j$.

As $(N'_{U'})^U = h_U * \mathbf{D}_E N_U = h_U * \mathbf{D}_E * h_U N_U = N_U$, there is exactly one index $j$ such that $A_j^U \ne 0$. For this index one has $m_j = 1$. By Proposition 3.1(3), there are smooth irreducible representations $W_1$ and $W_2$ of $G$ such that $W_1^{U'} \cong A_j$ and $W_2^{U'} \cong V_1^{U'}$. Since $W_1^U \cong W_2^U$, by Proposition 3.1(2), one has $W_1 \cong W_2$, so $A_j \cong V_1^{U'}$. Set $N_{U'} = A_j$.

Then $N := \bigcup_{U' \subset U} N_{U'}$ is a $E$-subrepresentation of $G$ in $V$ intersecting $V_0$ trivially. (The action of $\sigma \in G$ on $N_{U_L}$ for some finite extension $L$ of $F^U$ is given by the composition of the embedding $N_{U_L} \subseteq N_{U_{L\sigma(L)}}$ with $h_{L\sigma(L)} * \sigma * h_{L\sigma(L)} \in \mathrm{End}(N_{U_{L\sigma(L)}})$. Here $h_L := h_{U_L}$ is the Haar measure on $U_L$, and we have used $\sigma : W^{U_L} \longrightarrow W^{U_{\sigma(L)}} \subseteq W^{U_{L\sigma(L)}}$.) By Proposition 3.1(1), the $E$-representation $N$ of $G$ is irreducible. This contradicts the definition of $W_0$. $\square$

For any irreducible variety $Y$ over $k$ with $k(Y) = F^U$ for a compact open subgroup $U$ in $G$ one can identify the Hecke algebra $\mathcal{H}(U)$ with the $\mathbb{Q}$-algebra of non-degenerate correspondences on $Y$ (i.e., of formal linear combinations of $n$-subvarieties in $Y \times_k Y$ dominant over both factors $Y$). This follows from the facts that the set of double classes $U\backslash G/U$ can be identified with a basis of $\mathcal{H}(U)$ as a $\mathbb{Q}$-space via $[\sigma] \longmapsto h_U * \sigma * h_U$; that irreducible $n$-subvarieties in $Y \times_k Y$ dominant over both factors $Y$ are in a natural bijection with the set of maximal ideals in $F^U \otimes_k F^U$, and the following lemma.

**Lemma 3.3.** *Let $L, L' \subseteq F$ be field subextension of $k$ with* $\mathrm{tr.deg}(L/k) = q < \infty$. *Then the set of double classes $U_{L'}\backslash G/U_L$ is canonically identified with the set of all points in $\mathbf{Spec}(L \otimes_k L')$ of codimension $\ge q -$* $\mathrm{tr.deg}(F/L')$ *(so $U_{L'}\backslash G/U_L = \mathbf{Max}(L \otimes_k L')$, if $F = \overline{L'}$). Here $G/U_L = \{$embeddings of $L$ into $F$ over $k\}$.*

*Proof.* To any embedding $\sigma : L \hookrightarrow F$ over $k$ one associates the ideal in $L \otimes_k F$ generated by elements $x \otimes 1 - 1 \otimes \sigma x$ for all $x \in F$. It is maximal, since it is the kernel of the surjection $L \otimes_k F \xrightarrow{\sigma \cdot id} F$. Conversely, any maximal ideal $\mathfrak{m}$ in $L \otimes_k F$ determines a homomorphism $L \otimes_k F \xrightarrow{\varphi} (L \otimes_k F)/\mathfrak{m} = \Xi$ with $\Xi$ a field. Since its restriction to the subfield $k \otimes_k F$ is an embedding, one can regard $F$ as a subfield of $\Xi$. Let $t_1, \dots, t_m$ be a transcendence basis of $L$ over $k$. As $L$ is algebraic over $L_0 := k(t_1, \dots, t_m)$, $\Xi$ is integral over $\varphi(L_0 \otimes_k F)$, so the latter is a field. One has $\mathbf{Spec}(L_0 \otimes_k F) \subset \mathbf{Spec}(F[t_1, \dots, t_m]) = \mathbb{A}_F^m$. On any subvariety of $\mathbb{A}_F^m$ outside the union of all divisors defined over $k$ there is an $F$-point which also lies outside the union of all divisors defined over $k$, and therefore, $\mathbf{Max}(L_0 \otimes_k F) \subset \mathbf{Max}(F[t_1, \dots, t_m])$. This means that $\varphi(L_0 \otimes_k F) = F$, and therefore, $\Xi = F$. The restriction of $\varphi$ to $L \otimes_k k$ gives an embedding $\sigma : L \hookrightarrow F$ over $k$. $\square$

Let $U$ and $V$ be open compact subgroups of $G$, and $F^U = L$, $F^V = L'$. Then, in notation of §1.1, p.7, one has the isomorphism $h_V \mathbf{D}_{\mathbb{Q}} h_U \xrightarrow{\sim}$

$\mathrm{Hom}_G(\mathbf{D}_{\mathbb{Q}}h_V,\mathbf{D}_{\mathbb{Q}}h_U)$ given by right multiplication (the inverse sends a $G$-homomorphism to its value on $h_V$), which is evidently compatible with multiplication in $\mathbf{D}_{\mathbb{Q}}$ and composing $G$-homomorphisms

$$\mathrm{Hom}_G(\mathbf{D}_{\mathbb{Q}}h_W,\mathbf{D}_{\mathbb{Q}}h_V)\times\mathrm{Hom}_G(\mathbf{D}_{\mathbb{Q}}h_V,\mathbf{D}_{\mathbb{Q}}h_U)\longrightarrow\mathrm{Hom}_G(\mathbf{D}_{\mathbb{Q}}h_W,\mathbf{D}_{\mathbb{Q}}h_U).$$

The canonical projection $\mathbf{D}_{\mathbb{Q}}\longrightarrow\mathbb{Q}[G/V]$ identifies the $G$-module $\mathbf{D}_{\mathbb{Q}}h_V$ with $\mathbb{Q}[\{L'\overset{/k}{\hookrightarrow}F\}]$. The latter can be also regarded as the $G$-module of generic 0-cycles $\mathbb{Q}[\mathbf{Max}(L'\otimes_k F)]$. Similarly, $\mathbf{D}_{\mathbb{Q}}h_U=\mathbb{Q}[\{L\overset{/k}{\hookrightarrow}F\}]=\mathbb{Q}[\mathbf{Max}(L\otimes_k F)]$.

The correspondences $\mathbb{Q}[\mathbf{Max}(L\otimes_k L')]=\mathbb{Q}[V\backslash\{L\overset{/k}{\hookrightarrow}F\}]$ act on the space $\mathbb{Q}[\mathbf{Max}(L'\otimes_k F)]$ as follows. A correspondence $[\tau]\in V\backslash\{L\overset{/k}{\hookrightarrow}F\}$ sends a cycle $L'\overset{\sigma}{\hookrightarrow}F$ to $\sum\limits_{\xi\in V/U_{\tau(L)L'}}[\sigma\xi\tau]\in\mathbb{Q}[\{L\overset{/k}{\hookrightarrow}F\}]$.

This gives an isomorphism

$$\mathbb{Q}[\mathbf{Max}(L\otimes_k L')]\xrightarrow{\sim}\mathrm{Hom}_G(\mathbb{Q}[\{L'\overset{/k}{\hookrightarrow}F\}],\mathbb{Q}[\{L\overset{/k}{\hookrightarrow}F\}])$$

compatible with composing of correspondences and of $G$-homomorphisms. Its inverse is given by the composition

$$\begin{aligned}\mathrm{Hom}_G(\mathbb{Q}[\{L'\overset{/k}{\hookrightarrow}F\}],\mathbb{Q}[\{L\overset{/k}{\hookrightarrow}F\}])&\xrightarrow{(id_{L'})}\mathbb{Q}[\{L\overset{/k}{\hookrightarrow}F\}]\\ &\xrightarrow{\alpha}\mathbb{Q}[V\backslash\{L\overset{/k}{\hookrightarrow}F\}]=\mathbb{Q}[\mathbf{Max}(L\otimes_k L')],\end{aligned}$$

where $(id_{L'})$ is the value on the element $L'\overset{id}{\hookrightarrow}F$ and $\alpha([\tau])=\frac{1}{[L'\tau(L):L']}[\tau]$.

Let $A^q(Y)$ be the quotient of the $\mathbb{Q}$-space $Z^q(Y)$ of cycles on a smooth proper variety $Y$ over $k$ of codimension $q$ by the $\mathbb{Q}$-subspace $Z^q_{\sim}(Y)$ of cycles $\sim$-equivalent to zero for an adequate equivalence relation $\sim$. According to Hironaka, each smooth variety $X$ admits an open embedding $i$ into a smooth proper variety $\overline{X}$ over $k$. Then $A^q(-)$ can be extended to arbitrary smooth variety $X$ as the cokernel of the map $Z^q_{\sim}(\overline{X})\xrightarrow{i^*}Z^q(X)$ induced by restriction of cycles. This is independent of the choice of variety $\overline{X}$.[4]

In the standard way one extends the contravariant functors $A^q(\ )$ and $Z^q(\ )$ to contravariant functors on the category of smooth pro-varieties over $k$. Namely, if for a set of indices $I$, an inverse system $(X_j)_{j\in I}$ of smooth varieties over $k$ is formed with respect to flat morphisms and $X$ is the limit, then $Z^q(X)=\varinjlim_{j\in I}Z^q(X_j)$, where the direct system is formed with respect

[4] since for any pair of smooth compactifications $(\overline{X},\overline{X}')$ of $X$ there is their common refinement $\overline{X}\xleftarrow{\beta}\overline{X}''\xrightarrow{\beta'}\overline{X}'$, $i^*$ factors through $Z^q_{\sim}(\overline{X})\xrightarrow{\beta^*}Z^q_{\sim}(\overline{X}'')\xrightarrow{(i'')^*}Z^q(X)$ and $i^*Z^q_{\sim}(\overline{X})=(i'')^*Z^q_{\sim}(\overline{X}'')$.

to the pull-backs, and similarly for $A^q(\ )$. This is independent of the choice of the projective system defining $X$.

In particular, as for any commutative $k$-algebra $R$ the scheme $\mathbf{Spec}(R)$ is an inverse limit of a system of $k$-varieties, $A^q(R) := A^q(\mathbf{Spec}(R))$ is defined. Any automorphism $\alpha$ of the $k$-algebra $R$ induces a morphism of a system $(X_j)_{j\in I}$ defining $\mathbf{Spec}(R)$ to a system $(\alpha^*(X_j))_{j\in I}$ canonically equivalent to $(X_j)_{j\in I}$, and therefore, induces an automorphism of $A^q(Y_R)$ for any $k$-scheme $Y$. This gives a contravariant functor from a category of varieties over $k$ to the category of $\mathrm{Aut}(R/k)$-modules. Set $B^q(X) = A^q(X)$ for $\sim$=numerical equivalence.

In what follows $X$ will be of type $Y_F$ for a $k$-subscheme $Y$ in a variety over $k$. It should be stressed that in this case $B^q(X) = B^q(Y_F)$ means *not* some sort of numerical equivalence over $F$, but a limit of cycle groups modulo numerical equivalence of varieties over $k$.

The homomorphism of algebras $\mathcal{H}(U) \longrightarrow A^{\dim Y}(Y \times_k Y)$ is surjective for any smooth projective $Y$, as one can see from the following "moving lemma", applied in the case $X_1 = X_2 = Y$ and $Z = X_1 \times_k X_2$. (Its present form is suggested by the referee.)

**Lemma 3.4.** *Let $Z, X_1, \dots, X_r$ be irreducible projective varieties over $k$, and let $Z \xrightarrow{p_j} X_j$ be surjective maps. Let $\alpha \subset Z$ be an irreducible subvariety of dimension $q \ge \max\limits_{1\le j\le r} \dim X_j$. Then $\alpha$ is rationally equivalent to a linear combination of some irreducible subvarieties in $Z$ surjective (under maps $p_j$) over all $X_j$'s.*

*Proof.* Choose a closed irreducible subvariety $W$ in $Z$ containing $\alpha$ as a divisor such that $W$ is surjective (under maps $p_j$) over all $X_j$. We can replace $Z$ by $W$ and assume that $\alpha$ has codimension 1 in $Z$. Then we can replace $Z$ by the blowup of $Z$ along $\alpha$. Since $\alpha$ is the direct image of its pullback, we can assume $\alpha$ is a Cartier divisor on $Z$. Since our varieties are taken to be projective, any Cartier divisor is rationally equivalent to a difference of two very ample divisors, which we take to dominate the $X_j$'s. $\square$

**Corollary 3.5.** *The natural map $Z^q(k(X)\otimes_k k(Y)) \longrightarrow A^q(X\times_k Y)$ is surjective if $q \le \dim X \le \dim Y$; and $Z^q(k(Y) \otimes_k F) \longrightarrow A^q(Y_F)$ is surjective if $q \le \dim Y \le n$.* $\square$

**Proposition 3.6.** *Let $Y$ be a smooth irreducible proper variety over $k$ and $\dim Y \le n$. Let $X$ be a smooth variety over $k$. For each $q \ge 0$ there are canonical isomorphisms*

$$\begin{aligned} A^q(X_{k(Y)}) &\xrightarrow{\sim} \mathrm{Hom}_G(A^{\dim Y}(Y_F), A^q(X_F)) \\ &\xrightarrow{\sim} \mathrm{Hom}_G(Z^{\dim Y}(k(Y) \otimes_k F), A^q(X_F)). \end{aligned}$$

*For each pair of reduced irreducible group schemes $\mathcal{A}$ and $\mathcal{B}$ over $k$ there is a natural bijection*

$$\mathrm{Hom}(\mathcal{A},\mathcal{B}) := \mathrm{Hom}_{\text{group schemes}/k}(\mathcal{A},\mathcal{B}) \xrightarrow{\sim} \mathrm{Hom}_G(\mathcal{A}(F),\mathcal{B}(F)).$$

*Proof* of the first part uses Corollary 3.5, Lemma 3.3 and elementary intersection theory as follows. Each embedding $\sigma : k(Y) \overset{/k}{\hookrightarrow} F$ induces an identification $Z^{\dim Y}(k(Y)\otimes_k F) \xrightarrow{\sim} \mathbb{Q}[G/U_{\sigma(k(Y))}]$, and thus, for each $G$-module $M$ one has an isomorphism $M^{U_{\sigma(k(Y))}} \xrightarrow{\sim} \mathrm{Hom}_G(Z^{\dim Y}(k(Y)\otimes_k F), M)$ given by $m \longmapsto [\tau\sigma \longmapsto \tau m]$ for any $\tau \in G$.

Let $Z^{\dim Y}(k(Y)\otimes_k F) \xrightarrow{\varphi} A^q(X_F)$ be a $G$-homomorphism. Fix an embedding $k(Y) \overset{\sigma}{\hookrightarrow} F$. For any $\xi \in G$ one has $\varphi(\xi\sigma) = \xi\varphi(\sigma)$, in particular, if $\xi|_{\sigma(k(Y))} = id$ then $\varphi(\sigma) \in A^q(X_F)^{U_{\sigma(k(Y))}} = A^q(X_{\sigma(k(Y))})$.[5] From this and the fact that the pairing $A^q(X\times_k Y)\otimes A^{\dim Y}(Y_F) \longrightarrow A^q(X_F)$ factors through $A^q(X_{k(Y)})\otimes A^{\dim Y}(Y_F) \longrightarrow A^q(X_F)$,[6] one deduces that sending a cycle $\alpha \in A^q(X_{k(Y)})$ to the action of its arbitrary lifting to a correspondence on $X\times_k Y$ determines a homomorphism

$$A^q(X_{k(Y)}) \longrightarrow \mathrm{Hom}_G(A^{\dim Y}(Y_F), A^q(X_F)),$$

which is surjective and canonical. On the other hand, it has the inverse given by $\varphi \longmapsto \sigma_*^{-1}\varphi(\sigma)$, where $\sigma_* : A^q(X_{k(Y)}) \xrightarrow{\sim} A^q(X_{\sigma(k(Y))})$.

The map $\mathrm{Hom}(\mathcal{A},\mathcal{B}) \longrightarrow \mathrm{Hom}_G(\mathcal{A}(F),\mathcal{B}(F))$ is clearly injective. Let $\mathcal{A}(F) \xrightarrow{\varphi} \mathcal{B}(F))$ be a $G$-homomorphism. Fix an irreducible curve $C \subset \mathcal{A}$ over $k$ generating $\mathcal{A}$ as an algebraic group, i.e., with the dominant multiplication map $C^N \longrightarrow \mathcal{A}$ for any $N \geq \dim\mathcal{A}$. Fix a generic point $x \in C(F)$. Since $\varphi(\tau x) = \tau\varphi(x)$, the element $\varphi(x)$ is fixed by any element $\tau$ fixing $x$, so any coordinate of $\varphi(x)$ is a rational function over $k$ in coordinates of $x$, and therefore, this gives rise to a rational $k$-map $h : C \dashrightarrow \mathcal{B}$. Consider the rational $k$-map $C^N \dashrightarrow \mathcal{B}$ given by $(x_1,\dots,x_N) \longmapsto h(x_1)\cdots h(x_N)$. On the set of $F$-points out of the union of "vertical" divisors, (i.e., on $\mathbf{Max}(\underbrace{k(C)\otimes_k\cdots\otimes_k k(C)}_{N\ \text{copies}}\otimes_k F))$ this map coincides with one given by $(x_1,\dots,x_N) \longmapsto \varphi(x_1)\cdots\varphi(x_N) = \varphi(x_1\cdots x_N)$.

As the multiplication map $\mathbf{Max}(\underbrace{k(C)\otimes_k\cdots\otimes_k k(C)}_{N\ \text{copies}}\otimes_k F) \longrightarrow \mathcal{A}(F)$ is surjective for any integer $N \geq 2\dim\mathcal{A}$, the rational map $C^N \dashrightarrow \mathcal{B}$ factors through $C^N \dashrightarrow \mathcal{A} \xrightarrow{\widehat{h}} \mathcal{B}$. Since $\varphi$ is a homomorphism, $\widehat{h}$ should also be a homomorphism, and in particular, regular. $\square$

[5] For a subextension $L \subseteq F$ of $k$ let $L'$ be a purely transcendental subextension of $L$ over which $F$ is algebraic. By Galois descent property, $A^q(X_F)^{U_{L'}} = A^q(X_{L'})$. By the homotopy invariance, $A^q(X_{L'}) = A^q(X_L)$, so $A^q(X_L) \subseteq A^q(X_F)^{U_L} \subseteq A^q(X_F)^{U_{L'}} = A^q(X_{L'}) = A^q(X_L)$.

[6] since any correspondence supported on $X\times_k D$ for a divisor $D$ on $Y$ sends $A^{\dim Y}(Y_F)$ to zero

**Corollary 3.7.** *For any field $L'$ of finite type and of transcendence degree $m \le n$ over $k$, any field $L$ of finite type over $k$ and any integer $q \ge m$ there is a canonical isomorphism*

$$A^q(L \otimes_k L') \xrightarrow{\sim} \mathrm{Hom}_G(A^m(L' \otimes_k F), A^q(L \otimes_k F)), \tag{3.1}$$

*where the both groups are zero if $q > m$.*

*Proof.* Let $Y$ be an irreducible smooth projective variety over $k$ with the function field $k(Y) = L'$. The group $A^m(L' \otimes_k F)$ is the quotient of $A^m(Y_F)$ by the sum of the images of $A^{m-1}(\widetilde{D}_F)$ for all divisors $D$ on $Y$ and their desingularizations $\widetilde{D}$.

Then the target of (3.1) is a subgroup in $\mathrm{Hom}_G(A^m(Y_F), A^q(L \otimes_k F))$.

By Proposition 3.6, $\mathrm{Hom}_G(A^m(Y_F), A^q(L \otimes_k F)) = A^q(L \otimes_k L')$, and

$$\mathrm{Hom}_G(A^{m-1}(\widetilde{D}_F), A^q(L \otimes_k F)) = A^q(L \otimes_k k(D)),$$

which is zero, since $\dim D < q$. The vanishing of $\mathrm{Hom}_G(A^{m-1}(\widetilde{D}_F), A^q(L \otimes_k F))$ for all divisors $D$ on $Y$ implies the coincidence of both sides in (3.1). $\square$

**Proposition 3.8.** *The $G$-module*[7] *$\mathbf{B}^q_X = B^q(X \times_k F)$ is admissible for any smooth proper $k$-variety $X$ and any $q \ge 0$. If $n < \infty$ then $\mathbf{B}^q_X$ is semi-simple.*

*If $q = 0$, or $q = 1$, or $q = \dim X \le n$ then $\mathbf{B}^q_X$ is semi-simple and of finite length.*

*Proof.* By a standard argument, we may suppose that $k$ is embedded into the field of complex numbers $\mathbb{C}$, and thus, for any smooth proper $k$-variety $Y$ with $k(Y) \subset F$ the space $B^q(X \times_k F)^{U_{k(Y)}}$ is a quotient of the finite-dimensional space $Z^q(X \times_k Y)/\sim_{\mathrm{hom}} \subseteq H^{2q}((X \times_k Y)(\mathbb{C}), \mathbb{Q}(q))$, so the representation $B^q(X \times_k F)$ is admissible.

For each smooth irreducible variety $Y$ over $k$ with $k(Y) \subset F$ and $\overline{k(Y)} = F$ the kernel of $A^q(X \times_k Y) \longrightarrow A^q(X \times_k k(Y))$ is a $A^n(Y \times_k Y)$-submodule in $A^q(X \times_k Y)$, since for its arbitrary element $\alpha$ and for any element $\beta \in A^n(Y \times_k Y)$ one has $\alpha \circ \beta = \mathrm{pr}_{13*}(\mathrm{pr}_{12}^*\alpha \cdot \mathrm{pr}_{23}^*\beta)$, so the projection to $Y$ of the support of $\alpha \circ \beta$ is contained in $\mathrm{pr}_2((D \times_k Y) \bigcap \mathrm{supp}(\beta))$ for a divisor $D$ on $Y$, which is of dimension $n-1$, so cannot dominate $Y$. This implies that $A^q(X \times_k k(Y))$ has a natural structure of a $A^n(Y \times_k Y)$-module.

By [Jan], the algebra $B^n(Y \times_k Y)$ is semi-simple, so the $B^n(Y \times_k Y)$-module $B^q(X \times_k k(Y))$ is also semi-simple. By the moving lemma 3.4, the ring homomorphism $\mathcal{H}_{U_{k(Y)}} \longrightarrow B^n(Y \times_k Y)$, induced by the identification of the Hecke algebra $\mathcal{H}_{U_{k(Y)}}$ with the algebra of non-degenerate correspondences on $Y$, (see p.15) is surjective. This gives a (semi-simple) $\mathcal{H}_{U_{k(Y)}}$-module structure on any $B^n(Y \times_k Y)$-module. Then, by Lemma 3.2, the $G$-module $B^q(X \times_k F)$ is semi-simple.

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[7] Recall that $B^q(X \times_k F)$ is a limit of certain quotients of $\mathbb{Q}$-spaces of classes of numerical equivalence of cycles on smooth proper varieties over $k$, but not over $F$, cf. pp.16–17 before Lemma 3.4.

Now suppose that $n = \infty$. By the same result of Jannsen [Jan], the category of motives modulo numerical equivalence is semi-simple. Let $(X, \Delta_X) = \bigoplus_j (X, \pi_j)$ be a decomposition into a (finite) direct sum of irreducible submotives. Then $B^q(X_F) = \bigoplus_j \pi_j B^q(X_F)$. If $W = \pi_j B^q(X_F)$ is reducible there is a non-zero proper $G$-submodule $W_0$ in $W$. Fix elements $e_0 \in W_0 - 0$ and $e_1 \in W - W_0$. Then the common stabilizer of $e_0$ and $e_1$ is an open subgroup in $G$, so it contains a subgroup $U_L$ for a subfield $L$ of $F$ finitely generated over $k$. Let $F'$ be an algebraically closed extension of $L$ with $\dim X \leq \mathrm{tr.deg}(F'/k) < \infty$. Then $W_0^{G_{F/F'}}$ is a non-zero proper $G_{F'/k}$-module in $W^{G_{F/F'}} = \pi_j \left(B^q(X_F)\right)^{G_{F/F'}}$, so the length of the $G_{F'/k}$-module $\pi_j \left(B^q(X_F)\right)^{G_{F/F'}} = \pi_j B^q(X_{F'})$ is $\geq 2$. By Proposition 3.6, there is a canonical surjection $\mathrm{End}_{\mathrm{motive}/k}\left((X, \pi_j)\right) \longrightarrow \mathrm{End}_{G_{F'/k}}(\pi_j B^q(X_{F'}))$ if $q \in \{0, 1, \dim X\}$. By the irreducibility of $(X, \pi_j)$ we have a division algebra on the left hand side, but the algebra on the right hand side has divisors of zero since the $G_{F'/k}$-module $\pi_j B^q(X_{F'})$ is semi-simple, but not irreducible, giving contradiction.

Any cyclic semi-simple $G$-module, $\mathbf{B}_X^{\dim X}$ in particular (Corollary 3.5), is of finite length.

It follows from Lefschetz theorem on $(1,1)$-classes that for any smooth proper $k$-variety $Y$ one has $B^1(X \times_k Y) = B^1(X) \oplus \mathrm{Hom}(\mathrm{Alb}X, \mathrm{Alb}Y) \oplus B^1(Y)$. By Lefschetz hyperplane section theorem, inclusion $C \hookrightarrow X$ of any smooth 1-dimensional plane section $C$ of $X$ induces a surjection $\mathrm{Alb}C \longrightarrow \mathrm{Alb}X$. This implies that $\mathbf{B}_X^1$ is embedded into $B^1(X) \oplus \mathbf{B}_C^1$, so it is also of finite length. $\square$

**Corollary 3.9.** *One has* $\mathrm{Hom}_G(B^q(L' \otimes_k F), B^p(L \otimes_k F)) = 0$ *for any pair of fields* $L, L'$ *finitely generated over* $k$ *with* $\mathrm{tr.deg}(L/k) = p$, $\mathrm{tr.deg}(L'/k) = q$ *and* $p \neq q$.

*Proof.* If $p > n$, or $q > n$, then at least one of $B^q(L' \otimes_k F)$ and $B^p(L \otimes_k F)$ is zero, so we may assume that $\max(p, q) \leq n$. By Proposition 3.8, the $G$-modules $B^q(L' \otimes_k F)$ and $B^p(L \otimes_k F)$ are semi-simple, so $\mathrm{Hom}_G(B^q(L' \otimes_k F), B^p(L \otimes_k F))$ is isomorphic to $\mathrm{Hom}_G(B^p(L \otimes_k F), B^q(L' \otimes_k F))$, so we may assume that $p > q$. Then, by Corollary 3.7, one has $\mathrm{Hom}_G(B^q(L' \otimes_k F), B^p(L \otimes_k F)) = B^p(L \otimes_k L') = 0$. $\square$

**Corollary 3.10.** *Let* $X$ *and* $Y$ *be smooth irreducible proper varieties over* $k$. *Then the* $\mathbb{Q}$*-vector spaces* $B^{\dim X}(X_{k(Y)})$ *and* $B^{\dim Y}(Y_{k(X)})$ *are naturally dual. If* $n < \infty$ *and* $\dim X \leq n$ *this duality induces a non-degenerate* $G$*-equivariant pairing*

$$B^{\dim X}(X_F) \otimes \lim_{U \longrightarrow} B^n((Y_U)_{k(X)}) \longrightarrow \mathbb{Q}(\chi),$$

*where* $U$ *runs over the set of open compact subgroups in* $G$, $Y_U$ *is a smooth proper model of* $F^U$ *over* $k$ *(thus,* $\dim Y_U = n$*) and the direct system is formed with respect to the pull-backs on the cycles.*

*Proof.* Let $\dim Y \geq \dim X$. Set $n = \dim Y$. By Proposition 3.6,

$$B^{\dim X}(X_{k(Y)}) = \mathrm{Hom}_G(B^n(Y_F), B^{\dim X}(X_F))$$

and $B^n(Y_{k(X)}) = \mathrm{Hom}_G(B^{\dim X}(X_F), B^n(Y_F))$. By Proposition 3.8, the $G$-modules $B^n(Y_F)$ and $B^{\dim X}(X_F)$ are semi-simple and of finite length. For any $\alpha \in B^{\dim X}(X_{k(Y)})$ and $\beta \in B^n(Y_{k(X)})$ set $\langle \alpha \cdot \beta \rangle = \mathrm{tr}(\alpha \circ \beta)$ $(= \mathrm{tr}(\beta \circ \alpha))$. Here $\alpha$ and $\beta$ are considered as $G$-homomorphisms. If $\alpha \neq 0$ there is an element $\gamma \in B^n(Y_{k(X)})$ such that $\alpha \circ \gamma$ is a non-zero projector in $\mathrm{End}_G B^{\dim X}(X_F)$, so the form $\langle \quad \cdot \quad \rangle$ is non-degenerate.

The form $B^{\dim X}(X_F) \otimes \varinjlim_U B^n((Y_U)_{k(X)}) \longrightarrow \mathbb{Q}(\chi)$ is defined by $\alpha \otimes \beta \longmapsto \langle \alpha \cdot \beta \rangle \cdot [U]$, for any $\alpha \in B^{\dim X}(X_F)^U$ and $\beta \in B^n((Y_U)_{k(X)})$. By the projection formula, it is well-defined. □

The above examples of $G$-modules are obtained from some (pro-)varieties over $k$ by extending the base field to $F$. More generally, one can construct a $G$-module starting from some birationally invariant functor $\mathcal{F}$ on a category of varieties over $k$, or on a category of field extensions of $k$ (as in Corollary 3.10).

Starting with the functor $\mathrm{Div}_{\mathrm{alg}}$ of algebraically trivial divisors on the category of smooth proper varieties over $k$,[8] or with the functor $\mathrm{Pic}^\circ_{\mathbb{Q}}$, we get another examples of $G$-modules of this type: $\mathrm{Div}^\circ_{\mathbb{Q}} = \varinjlim_U \mathrm{Div}_{\mathrm{alg}}(Y_U)_{\mathbb{Q}}$, and $\mathrm{Pic}^\circ_{\mathbb{Q}} = \varinjlim_U \mathrm{Pic}^\circ(Y_U)_{\mathbb{Q}}$, where $U$ runs over the set of open subgroups of type $U_L$ and $Y_U$ is a smooth projective model of $F^U = L$ over $k$.

If $\mathcal{A}$ is a commutative group scheme over $k$, we set $W_{\mathcal{A}} = \mathcal{A}(F)/\mathcal{A}(k)$.

**Proposition 3.11.** $\mathrm{Pic}^\circ_{\mathbb{Q}} = \bigoplus_{\mathcal{A}} \mathcal{A}(k) \otimes_{\mathrm{End}(\mathcal{A})} W_{\mathcal{A}^\vee}$, *where $\mathcal{A}$ runs over the isogeny classes of simple abelian varieties over $k$, and $\mathcal{A}^\vee := \mathrm{Pic}^\circ \mathcal{A}$ is the dual abelian variety.*

*Proof.* For any open compact subgroup $U$ there is a canonical decomposition

$$\bigoplus_{\mathcal{A}} \mathcal{A}(k) \otimes_{\mathrm{End}(\mathcal{A})} \mathrm{Hom}(\mathcal{A}, \mathrm{Pic}^\circ Y_U)_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Pic}^\circ(Y_U)_{\mathbb{Q}} = \left(\mathrm{Pic}^\circ_{\mathbb{Q}}\right)^U$$

given by $a \otimes \varphi \longmapsto \varphi(a)$ for any $a \in \mathcal{A}(k)$ and any $\varphi \in \mathrm{Hom}(\mathcal{A}, \mathrm{Pic}^\circ Y_U)_{\mathbb{Q}}$, where $\mathcal{A}$ runs over the isogeny classes of simple abelian varieties over $k$. (Clearly, the image of $a \otimes t\varphi$, i.e., $t\varphi(a) := \varphi(ta)$ coincides with the image of $ta \otimes \varphi$ for any $t \in \mathrm{End}(\mathcal{A})$, so the map is well-defined.) Passing to the direct limit with respect to $U$, we get $\mathrm{Pic}^\circ_{\mathbb{Q}} = \bigoplus_{\mathcal{A}} \left(\mathrm{Pic}^\circ_{\mathbb{Q}}\right)_{\mathcal{A}}$, where $\left(\mathrm{Pic}^\circ_{\mathbb{Q}}\right)_{\mathcal{A}} := \mathcal{A}(k) \otimes_{\mathrm{End}(\mathcal{A})} \varinjlim_U \mathrm{Hom}(\mathcal{A}, \mathrm{Pic}^\circ Y_U)_{\mathbb{Q}}$ is the $\mathcal{A}$-isotypic component.

[8] $\mathrm{Div}_{\mathrm{alg}}(Y_U)$ is independent of the choice of $Y_U$, since any birational morphism induces a homomorphism of the groups of algebraically trivial divisors, which is an isomorphism of the subgroups of linearly trivial divisors (= multiplicative groups of the function fields modulo $k^\times$) and induces an isomorphism of the quotients modulo linear equivalence (= $\mathrm{Pic}^\circ$-groups).

Using the identifications

$$\mathrm{Hom}(\mathcal{A},\mathrm{Pic}^{\circ}Y_U)_{\mathbb{Q}}=\mathrm{Hom}(\mathrm{Alb}Y_U,\mathcal{A}^{\vee})_{\mathbb{Q}}$$
$$=(\mathrm{Mor}(Y_U,\mathcal{A}^{\vee})/\mathcal{A}^{\vee}(k))_{\mathbb{Q}}=(W_{\mathcal{A}^{\vee}})^U,$$

we get $\lim\limits_{U\longrightarrow}\mathrm{Hom}(\mathcal{A},\mathrm{Pic}^{\circ}Y_U)_{\mathbb{Q}}=W_{\mathcal{A}^{\vee}}$, so

$$\left(\mathrm{Pic}^{\circ}_{\mathbb{Q}}\right)_{\mathcal{A}}=\mathcal{A}(k)\otimes_{\mathrm{End}(\mathcal{A})}W_{\mathcal{A}^{\vee}}. \qquad \square$$

**Corollary 3.12.** *For any smooth irreducible variety $X$ of dimension $\le n+1$ over $k$ and $q\in\{0,1,2,\dim X\}$ there is a unique $G$-submodule in $\mathbf{B}^q_X=B^q(X\times_k F)$ isomorphic to $B^q(k(X)\otimes_k F)$.*

*Proof.* For any $q\ge 0$ and any adequate relation $\sim$ one has the short exact sequence

$$\bigoplus_{D\in X^1}A^{q-1}(\widetilde{D}_F)\longrightarrow A^q(X_F)\longrightarrow A^q(k(X)\otimes_k F)\longrightarrow 0.$$

Let $F'$ be an algebraically closed extension of $k$ in $F$ with $\dim X-1\le \mathrm{tr.deg}(F'/k)<\infty$. As $(\mathbf{B}^q_X)^{G_{F/F'}}$ is semi-simple, there is a subrepresentation of $G_{F'/k}$ in $(\mathbf{B}^q_X)^{G_{F/F'}}$ isomorphic to $B^q(k(X)\otimes_k F')$.

By Proposition 3.6, there is an embedding

$$\mathrm{Hom}_{G_{F'/k}}(A^{q-1}(\widetilde{D}_{F'}),A^q(k(X)\otimes_k F'))\hookrightarrow A^q(k(X)\otimes_k k(D))=0$$

for $q=\dim X$; $\mathrm{Hom}_{G_{F'/k}}(A^{q-1}(\widetilde{D}_{F'}),A^q(k(X)\otimes_k F'))=0$ for $q\in\{0,1\}$; and $A^1(\widetilde{D}_{F'})$ is isomorphic to a subquotient of $A^1(C_{F'})\oplus\mathbb{Q}^N$ for a smooth proper curve $C$, so

$$\mathrm{Hom}_{G_{F'/k}}(A^1(\widetilde{D}_{F'}),A^2(k(X)\otimes_k F'))\subseteq A^2(k(X)\otimes_k k(C))=0.$$

This implies that for any $q$ in the range of the statement, any $G_{F'/k}$-equivariant homomorphism

$$(\mathbf{B}^q_X)^{G_{F/F'}}\longrightarrow B^q(k(X)\otimes_k F')$$

factors through an endomorphism of $B^q(k(X)\otimes_k F')$, and therefore, by semi-simplicity, that there is a unique $G_{F'/k}$-submodule in $(\mathbf{B}^q_X)^{G_{F/F'}}$ isomorphic to $B^q(k(X)\otimes_k F')$. Then the union over all $F'$ of such $G_{F'/k}$-submodules is the unique $G$-submodule in $\mathbf{B}^q_X$ isomorphic to $B^q(k(X)\otimes_k F)$. $\square$

For each open compact subgroup $U\subset G$ and a smooth irreducible variety $Y$ over $k$ with $k(Y)=F^U$ we define a semi-simple $G$-module (of finite length) $\mathbf{B}^q_{Z,Y}$ as the minimal one such that the $\mathcal{H}(U)$-module $(\mathbf{B}^q_{Z,Y})^U$ is isomorphic to $B^q(Z\times_k Y)$. By Proposition 3.1, it exists and it is unique.

**Lemma 3.13.** *Let $X$, $Y$ and $Z$ be smooth irreducible $k$-varieties,* $\dim X = \dim Y = n \geq \dim Z$, *and $p, q \geq 0$ integers. Let*

$$\mathcal{B}^{p,q}_{X,Y,Z} = \mathrm{Hom}_{\mathcal{H}(U)}(B^q(Z \times_k Y), B^p(k(X) \otimes_k k(Y))).$$

*Then $\mathcal{B}^{p,q}_{X,Y,Z} = 0$, if either $q = \dim Z < p$, or $q = n$ and $\dim Z < p$, or $q > n$ and $p + q > \dim Z + n$, or $q < p$ and $q \in \{0, 1\}$.*

*Proof.*

- Let $\dim Z = q < p$. As $B^q(Z \times_k Y) = (\mathbf{B}^q_{Z,Y})^U$, it follows from the moving lemma 3.4 that $W_1 := \mathbf{B}^q_{Z,Y}$ is a quotient of the module $Z^q(k(Z) \otimes_k F)$ (since $W_1^U = W^U$, where $W$ is the quotient of $Z^q(k(Z) \otimes_k F)$ by its $G$-submodule generated by the kernel of $Z^q(k(Z) \otimes_k k(Y)) \longrightarrow B^q(Z \times_k Y)$).
  As $\mathrm{Hom}_G(Z^q(k(Z) \otimes_k F), B^p(k(X) \otimes_k F)) = 0$, this gives
  $$\mathrm{Hom}_G(W_1, B^p(k(X) \otimes_k F)) = 0.$$
  We need to show that $\mathrm{Hom}_{\mathcal{H}(U)}(W^U, B^p(k(X) \otimes_k k(Y))) = 0$.
  By Proposition 3.1, for any pair $(W_1, W_2)$ of semi-simple $G$-modules the natural homomorphism $\mathrm{Hom}_G(W_1, W_2) \longrightarrow \mathrm{Hom}_{\mathcal{H}(U)}(W_1^U, W_2^U)$ is surjective. Then, as its source is zero when $W_2 = B^p(k(X) \otimes_k F)$, we get the vanishing of the space $\mathrm{Hom}_{\mathcal{H}(U)}((\mathbf{B}^q_{Z,Y})^U, B^p(k(X) \otimes_k k(Y)))$.
- Let $c = n - \dim Z$, so the variety $Z \times \mathbb{P}^c$ is $n$-dimensional. As there is an embedding of $\mathcal{H}(U)$-modules $B^n(Z \times_k Y) \hookrightarrow B^n(Z \times_k \mathbb{P}^c \times_k Y)$, it is enough to show the vanishing of $\mathcal{B}^{p,n}_{X,Y,Z\times\mathbb{P}^c}$ for $p > \dim Z$. By semi-simplicity, the latter is a subgroup in $\mathrm{Hom}_G(\mathbf{B}^n_{Z\times\mathbb{P}^c,Y}, B^p(k(X)\otimes_k F)) \subseteq \mathrm{Hom}_G(Z^n(k(Z \times \mathbb{P}^c) \otimes_k F), B^p(k(X) \otimes_k F))$. By Proposition 3.6, the latter coincides with $B^p(k(X) \otimes_k k(Z \times \mathbb{P}^c))$, which is dominated by $B^p(\mathbb{A}^c_{k(X)\otimes_k k(Z)}) \xleftarrow{\sim} B^p(k(X) \otimes_k k(Z)) = 0$, and thus, $\mathcal{B}^{p,n}_{X,Y,Z} = 0$, if $p > \dim Z$.
- As the $\mathcal{H}(U)$-module $\bigoplus_{D \in Z^{q-n}} B^n(\widetilde{D} \times_k Y)$ surjects onto the $\mathcal{H}(U)$-module $B^q(Z \times_k Y)$ when $q > n$, the vanishing of $\mathcal{B}^{p,q}_{X,Y,Z}$ follows from $\mathcal{B}^{p,n}_{X,Y,\widetilde{D}} = 0$ for each subvariety $D$ of codimension $q - n$ on $Z$.
- If $q = 1$ then $B^1(Z \times_k Y)$ is a subquotient of $B^1(C \times_k Y)$ for a smooth curve $C$, so we are reduced to the case $q = \dim Z < p$. The case $q = 0$ is trivial. $\square$

**Proposition 3.14.** *Let $X$ and $Y$ be smooth irreducible varieties over $k$, and either $q \in \{0, 1, 2\}$, or $q = \dim X = \dim Y$. Then there is a unique submodule in $B^q(X \times_k Y)$ over $\left(B^{\dim X}(X \times_k X) \otimes B^{\dim Y}(Y \times_k Y)^{\mathrm{op}}\right)$ isomorphic to its quotient $B^q(k(X) \otimes_k k(Y))$.*

*Proof.* The existence of such submodule follows from the semi-simplicity of $B^q(X \times_k Y)$ ([Jan]). By Lemma 3.13,

$$\mathrm{Hom}_{B^{\dim Y}(Y \times_k Y)}(B^{q-1}(\widetilde{D} \times_k Y), B^q(k(X) \otimes_k k(Y))) = 0.$$

As the kernel of the projection $B^q(X \times_k Y) \longrightarrow B^q(k(X) \otimes_k k(Y))$ is generated by the images of $B^{q-1}(X \times_k \widetilde{E})$ and $B^{q-1}(\widetilde{D} \times_k Y)$ for all divisors $D$ on $X$ and all divisors $E$ on $Y$, this implies the uniqueness. $\square$

*3.1 The projector $\Delta_{k(X)}$*

For any pair of varieties $X, Y$ let ${}^t$ be the transposition of cycles, induced by $X \times Y \xrightarrow{\sim} Y \times X$. Denote by $\Delta_{k(X)} = {}^t\Delta_{k(X)}$ the identity (diagonal) element in $B^n(k(X) \otimes_k k(X))$ considered as an element of $B^n(X \times_k X)$.

**Lemma 3.15.** *For any irreducible smooth proper $k$-variety $X$ of dimension $n$ the element $\Delta_{k(X)}$ is a central projector in the algebra $B^n(X \times_k X)$. The left (equivalently, right) ideal generated by $\Delta_{k(X)}$ coincides with (the image of) $B^n(k(X) \otimes_k k(X))$.*

*Proof.* Denote by $\varphi$ the projection $B^n(X \times_k X) \longrightarrow B^n(k(X) \otimes_k k(X))$ and by $\psi$ its unique section $B^n(k(X) \otimes_k k(X)) \longrightarrow B^n(X \times_k X)$. The kernel of $\varphi$ coincides with the sum of the kernels of $B^n(X \times_k X) \longrightarrow B^n(X \times_k k(X))$ and of $B^n(X \times_k X) \longrightarrow B^n(k(X) \times_k X)$, where the projections are induced by the ring homomorphisms

$$B^n(X \times_k X) \longrightarrow \mathrm{End}_G B^n(X \times_k F) = B^n(X \times_k k(X))$$

and $B^n(X \times_k X) \longrightarrow \mathrm{End}_G B^n(F \times_k X) = B^n(k(X) \times_k X)$, so $\ker \varphi$ is a two-sided ideal. Then the image of $\varphi$ is a $B^n(X \times_k X)$-bi-module, and thus, the image of $\psi$ is a two-sided ideal in $B^n(X \times_k X)$.

As $\alpha \circ \Delta_X = \Delta_X \circ \alpha$ for any $\alpha \in B^n(X \times_k X)$, and $\varphi$ and $\psi$ are morphisms of $B^n(X \times_k X)$-bi-modules, one has $\alpha\psi(\varphi(\Delta_X)) = \psi(\varphi(\alpha \circ \Delta_X)) = \psi(\varphi(\Delta_X \circ \alpha)) = \psi(\varphi(\Delta_X))\alpha$, so $\alpha\Delta_{k(X)} = \Delta_{k(X)}\alpha$. The $B^n(X \times_k X)$-action on $B^n(k(X) \otimes_k k(X))$ factors through $B^n(k(X) \otimes_k k(X))$, so $\Delta_{k(X)}^2 = \Delta_X \Delta_{k(X)} = \Delta_{k(X)}$. $\square$

**Lemma 3.16.** $\Delta_{k(X)} B^{\dim X}(X_L) = B^{\dim X}(k(X) \otimes_k L)$ *for any irreducible smooth proper $k$-variety $X$ and any field extension $L$ of $k$.*

*Proof.* Set $d = \dim X$. Using homotopy invariance and the Galois descent property of $B^*$, we may replace $L$ by an algebraically closed extension $F$ with $\mathrm{tr.deg}(F/k) = n \geq d$ and then $\Delta_{k(X)} B^d(X_L) = (\Delta_{k(X)} B^d(X_F))^{G_{F/L}}$ and $B^d(k(X) \otimes_k L) = B^d(k(X) \otimes_k F)^{G_{F/L}}$.

By Proposition 3.6 and Corollary 3.7, the canonical maps

$$\begin{aligned} B^d(k(X) \otimes_k k(X)) &\longrightarrow \mathrm{End}_G B^d(k(X) \otimes_k F) \\ &\longrightarrow \mathrm{Hom}_G(B^d(X_F), B^d(k(X) \otimes_k F)) \end{aligned}$$

are isomorphisms. As $\Delta_{k(X)}$ is the identity element in $\mathrm{End}_G B^d(k(X) \otimes_k F)$, this means that $\Delta_{k(X)} B^d(X_F) = B^d(k(X) \otimes_k F)$. $\square$

**Proposition 3.17.** $(X, \Delta_{k(X)})$ *is the maximal primitive $n$-submotive of the motive $(X, \Delta_X)$ for any smooth irreducible proper $n$-dimensional $k$-variety $X$. The motive $(X, \Delta_{k(X)})$ is a birational invariant of $X$.*

*Proof.* To show that $(X, \Delta_{k(X)})$ is a primitive $n$-motive, we have to check that the $\mathbb{Q}$-vector space $W := \Delta_{k(X)} B^n(X \times_k Y \times \mathbb{P}^1)$ is zero for any variety $Y$ of dimension $< n$. Replacing $Y$ with $Y \times \mathbb{P}^{n-\dim Y-1}$ we may suppose that $\dim Y = n-1$.

$W$ is a left $B^n(k(X) \otimes_k k(X))$-module, since by Lemma 3.15, $\Delta_{k(X)}$ is a central projector in $B^n(X \times_k X)$ and $B^n(X \times_k X)\Delta_{k(X)} = B^n(k(X) \otimes_k k(X))$.

In notations of Lemma 3.13 one has

$$\begin{aligned}\mathrm{Hom}_{B^n(X\times_k X)}(B^n(X \times_k Y \times \mathbb{P}^1), B^n(k(X) \otimes_k k(X))) &=: \mathcal{B}^{n,n}_{X,X,Y\times\mathbb{P}^1} \\ &= \mathcal{B}^{n,n}_{X,X,Y} \bigoplus \mathcal{B}^{n,n-1}_{X,X,Y}.\end{aligned}$$

By the first case of Lemma 3.13, one has $\mathcal{B}^{n,n-1}_{X,X,Y} = 0$; by the second case of Lemma 3.13, one has $\mathcal{B}^{n,n}_{X,X,Y} = 0$, so $\mathcal{B}^{n,n}_{X,X,Y\times\mathbb{P}^1} = 0$. Then the semi-simplicity implies that

$$\mathrm{Hom}_{B^n(X\times_k X)}(B^n(k(X) \otimes_k k(X)), B^n(X \times_k Y \times \mathbb{P}^1)) = 0.$$

As $W$ is a quotient of a direct sum of several copies of $B^n(k(X)\otimes_k k(X))$, but there are no non-zero $B^n(X \times_k X)$-module quotients of $B^n(k(X) \otimes_k k(X))$ in $B^n(X \times_k Y \times \mathbb{P}^1)$, this means that $W = 0$.

For the maximality of $(X, \Delta_{k(X)})$ among primitive $n$-submotives of the motive $(X, \Delta_X)$, we have to show that $\mathrm{Hom}(M, X) = \mathrm{Hom}(M, (X, \Delta_{k(X)}))$ for any primitive $n$-motive $M = (Z, \pi)$. $\mathrm{Hom}(X, M) = \pi B^n(Z \times_k X)$. For any divisor $D$ on $X$ one has $\pi B^{n-1}(Z \times_k \widetilde{D}) = 0$, so from the exact sequence

$$\bigoplus_{D\in X^1} \pi B^{n-1}(Z \times_k \widetilde{D}) \longrightarrow \pi B^n(Z \times_k X) \longrightarrow \pi B^n(Z \times_k k(X)) \longrightarrow 0$$

we get $\pi B^n(Z \times_k X) = \pi B^n(Z \times_k k(X))$. By Proposition 3.6,

$$B^n(Z \times_k k(X)) = \mathrm{Hom}_G(B^n(X_F), B^n(Z_F)),$$

so $\mathrm{Hom}(X, M) = \pi\mathrm{Hom}_G(B^n(X_F), B^n(Z_F))$. By Proposition 3.6, for any divisor $D$ on $X$ one has

$$\mathrm{Hom}_G(B^{n-1}(D_F), \pi B^n(Z_F)) = \pi B^{n-1}(Z \times_k k(D)) = 0,$$

so $\mathrm{Hom}(X, M) = \pi\mathrm{Hom}_G(B^n(k(X) \otimes_k F), B^n(Z_F))$. By Lemma 3.16, the space $B^n(k(X)\otimes_k F)$ coincides with the image of the projector $\Delta_{k(X)}$ acting on $B^n(X_F)$, so

$$\begin{aligned}\mathrm{Hom}(X, M) &= \pi\mathrm{Hom}_G(\Delta_{k(X)} B^n(X_F), B^n(Z_F)) \\ &= \pi\mathrm{Hom}_G(B^n(X_F), B^n(Z_F))\Delta_{k(X)}.\end{aligned}$$

Since the space $\mathrm{Hom}_G(B^n(X_F), B^n(Z_F))$ is a quotient of $B^n(Z \times_k X)$, we get that the space $\mathrm{Hom}(X, M)$ is a quotient of $\pi B^n(Z \times_k X)\Delta_{k(X)} = \mathrm{Hom}((X, \Delta_{k(X)}), M)$. On the other hand, the space $\mathrm{Hom}((X, \Delta_{k(X)}), M)$

is a quotient of $\mathrm{Hom}(X,M)$. As both spaces are finite-dimensional, they coincide.

The birational invariantness of $(X,\Delta_{k(X)})$ follows from the birational invariantness of $Y^{\mathrm{prim}}$ for any smooth projective variety $Y$ over $k$ explained in the beginning of §3.2 below.[9] $\square$

**Corollary 3.18.** *Let $X$ and $Y$ be smooth irreducible proper varieties over $k$, and $\dim X=\dim Y=n$. Then the unique submodule of the module $B^n(X\times_k Y)$ over $\left(B^{\dim X}(X\times_k X)\otimes B^{\dim Y}(Y\times_k Y)^{\mathrm{op}}\right)$ isomorphic to its quotient $B^n(k(X)\otimes_k k(Y))$ coincides with $\Delta_{k(X)}\cdot B^n(X\times_k Y)=B^n(X\times_k Y)\cdot\Delta_{k(Y)}=\Delta_{k(X)}\cdot B^n(X\times_k Y)\cdot\Delta_{k(Y)}$.*

*Proof.* By Proposition 3.17, $(X,\Delta_{k(X)})$ and $(Y,\Delta_{k(Y)})$ are the maximal primitive $n$-submotives in $(X,\Delta_X)$ and $(Y,\Delta_Y)$, so

$$\mathrm{Hom}(X,(Y,\Delta_{k(Y)}))=\mathrm{Hom}((X,\Delta_{k(X)}),(Y,\Delta_{k(Y)})).$$

By definition,

$$\begin{aligned}&\mathrm{Hom}(X,(Y,\Delta_{k(Y)})):=B^n(X\times_k Y)\cdot\Delta_{k(Y)}\\ &\quad\text{and}\quad \mathrm{Hom}((X,\Delta_{k(X)}),(Y,\Delta_{k(Y)})):=\Delta_{k(X)}\cdot B^n(X\times_k Y)\cdot\Delta_{k(Y)},\end{aligned}$$

and thus, $B^n(X\times_k Y)\cdot\Delta_{k(Y)}=\Delta_{k(X)}\cdot B^n(X\times_k Y)\cdot\Delta_{k(Y)}$.

Similarly, $\Delta_{k(X)}\cdot B^n(X\times_k Y)=\Delta_{k(X)}\cdot B^n(X\times_k Y)\cdot\Delta_{k(Y)}$.

As in the proof of maximality of $(X,\Delta_{k(X)})$ in Proposition 3.17 we have $\Delta_{k(X)}\cdot B^n(X\times_k Y)=\Delta_{k(X)}\cdot B^n(X\times_k k(Y))$. By Lemma 3.16 the latter coincides with $B^n(k(X)\otimes_k k(Y))$. $\square$

### *3.2 The functors $\mathbb{B}^\bullet$ and $\mathfrak{B}^q$*

For a smooth projective variety $Y$ over $k$ let $Y^{\mathrm{prim}}$ be the motive defined by $Y^{\mathrm{prim}}:=\bigcap_{Y\xrightarrow{\varphi}M\otimes\mathbb{L}}\ker\varphi$, where $M$ runs over isomorphism classes of *effective* motives, or equivalently,

$$Y^{\mathrm{prim}}:=\mathrm{coker}\left[\bigoplus_{M\otimes\mathbb{L}\xrightarrow{\varphi}Y}M\otimes\mathbb{L}\longrightarrow Y\right].$$

Clearly, $Y\longmapsto Y^{\mathrm{prim}}$ is a functor from the category of smooth projective varieties to the category of motives modulo numerical equivalence. Any birational map is a composition of blow-ups and blow-downs with smooth centers ([AKMW, W]). As a blow-up does not change $Y^{\mathrm{prim}}$ (cf. [M]), this implies that $Y^{\mathrm{prim}}$ is an invariant of the function field $k(Y)$. According to

[9] One can show this directly as follows. By Corollary 3.18, for any primitive $n$-motive $M\cong(Y,\pi)$ with $\dim Y=n$ one has $\mathrm{Hom}(X,M):=\pi B^n(Y\times_k X)=\pi B^n(k(Y)\times_k k(X))$, which is independent of the model of $k(X)$.

Hironaka, for any subfield $L$ of $F$ finitely generated over $k$ there exists a smooth projective variety $Y_{[L]}$ over $k$ with the function field $L$, and therefore, one gets a canonical projective system of motives $\{Y_{[L]}^{\mathrm{prim}}\}_L$ indexed by subfields $L$ of $F$ finitely generated over $k$.

Now we define the functor $\mathbb{B}^{\bullet} = \bigoplus \mathbb{B}^{[i]}$ of Theorem 1.1 from the category of motives modulo numerical equivalence to the category of graded $\mathbb{Q}$-spaces by setting $\mathbb{B}^{[i]} = \lim\limits_{L \longrightarrow} \mathrm{Hom}\left(Y_{[L]}^{\mathrm{prim}} \otimes \mathbb{L}^{\otimes i}, -\right)$ for its component of degree $i$. Let also $\mathfrak{B}^q$ denotes the restriction of $\mathbb{B}^{[0]}$ to the subcategory of the primitive $q$-motives. $G$ acts on the projective system $\{Y_{[L]}^{\mathrm{prim}}\}_L$ by $Y_{[L]} \xrightarrow{\sigma} Y_{[\sigma(L)]}$, $\sigma(L) \xrightarrow{\sigma^{-1}} L$, so $G$ acts on the limits $\mathfrak{B}^q(M)$ and $\mathbb{B}^{\bullet}(M)$.

Remark. Any Grothendieck motive modulo numerical equivalence $M = (X, \pi)$ is isomorphic to $\bigoplus\limits_{0 \le i, j, i+j \le \dim X} M_{ij} \otimes \mathbb{L}^{\otimes i}$, where $M_{ij}$ is a primitive $j$-motive and $\mathbb{L} = (\mathbb{P}^1, \mathbb{P}^1 \times \{0\})$, so $\mathbb{B}^{[i]}(M) \cong \bigoplus_j \mathfrak{B}^j(M_{ij})$. One proves this by induction on dimension $d$ of $X$ as follows. Let $M_{0d} = \bigcap_{\varphi} \ker(\varphi)$, where $\varphi$ runs over the set of morphisms from $M$ to motives of type $(Y \times \mathbb{P}^1, \Delta)$ for all $Y$ with $\dim Y < d$. (By Proposition 3.17, $M_{0d} = (X, \pi \circ \Delta_{k(X)})$.) As the length of $M$ is $\le \dim_{\mathbb{Q}} \mathrm{End}(M) < \infty$, the motive $M/M_{0d}$ can be embedded into a finite direct sum of $(Y_j \times \mathbb{P}^1, \Delta)$ with $\dim Y_j < d$. As $(Y_j \times \mathbb{P}^1, \Delta) = (Y_j, \Delta) \oplus (Y_j, \Delta) \otimes \mathbb{L}$, the induction is completed. In fact, the decomposition $M = \bigoplus\limits_{0 \le i, j, i+j \le \dim X} \widetilde{M}_{ij}$, where $\widetilde{M}_{ij}$ is isomorphic to $M_{ij} \otimes \mathbb{L}^{\otimes i}$, is canonical since

$$\widetilde{M}_{ij} = \mathrm{Im} \left[ \bigoplus_{N:\text{primitive } j\text{-motives}, N \otimes \mathbb{L}^{\otimes i} \xrightarrow{\varphi} M} N \otimes \mathbb{L}^{\otimes i} \xrightarrow{\sum \varphi} M \right].$$

**Proposition 3.19.** *If $\dim X = q \le n$ and $M = (X, \pi)$ is a primitive $q$-motive then $\mathfrak{B}^q(M) = \pi B^q(X_F)$.*

*Proof.* First we wish to show that $\mathbb{B}^{[0]}(M)^{U_L} = \mathrm{Hom}\left(Y_{[L]}, M\right)$ for any $L \subset F$ of finite type over $k$, so (by Lemma 6.1 below) $\mathbb{B}^{[0]}(M)^{G_{F/K}} = \lim\limits_{K \supseteq L \rightarrow} \mathrm{Hom}\left(Y_{[L]}, M\right)$. Any element $\alpha \in \mathbb{B}^{[0]}(M)^{U_L}$ belongs to the image of $\mathrm{Hom}\left(Y_{[L'']}, M\right)$ for some $L'' \supseteq L$. We may assume that $L''$ is a Galois extension of a purely transcendental extension $L'$ of $L$. Fix a finite affine open covering $\{U_\gamma\}$ of $Y_{[L']}$. Let $B_\gamma$ be the integral closure of $\mathcal{O}(U_\gamma)$ in $L''$, $V_\gamma = \mathbf{Spec}(B_\gamma)$ and $Y'' = \coprod_{\gamma} {}_{Y_{[L']}} V_\gamma$. Then $\mathrm{Gal}(L''/L')$ acts on each $V_\gamma$, and therefore, $\mathrm{Gal}(L''/L')$ acts on $Y''$ with a smooth quotient $Y_{[L']}$, so $Y''$ is projective. By equivariant version of resolution of singularities, there is a smooth projective variety $Y_{[L'']}$ with a $\mathrm{Gal}(L''/L')$-action and a $\mathrm{Gal}(L''/L')$-equivariant birational morphism $Y_{[L'']} \longrightarrow Y''$, so $\alpha$ belongs to the image of $\mathrm{Hom}\left(Y_{[L'']}, M\right)^{\mathrm{Gal}(L''/L')} = \pi B^q\left(X \times_k Y_{[L'']}\right)^{\mathrm{Gal}(L''/L')}$. Let

$\widetilde{Y}$ be a smooth projective variety admitting birational morphisms to $Y_{[L']}$ and to $Y_{[L'']}/\mathrm{Gal}(L''/L')$. Then

$$\begin{aligned}\mathrm{Hom}\left(Y_{[L']}^{\mathrm{prim}}, M\right) &= \mathrm{Hom}\left(\widetilde{Y}^{\mathrm{prim}}, M\right) = \pi B^q\left(X \times_k \widetilde{Y}\right) \\ &\longrightarrow\!\!\!\!\rightarrow \pi B^q\left(X \times_k Y_{[L'']}/\mathrm{Gal}(L''/L')\right) = \pi B^q\left(X \times_k Y_{[L'']}\right)^{\mathrm{Gal}(L''/L')}.\end{aligned}$$

On the other hand, by the projection formula,

$$\mathrm{Hom}\left(Y_{[L']}^{\mathrm{prim}}, M\right) = \pi B^q\left(X \times_k Y_{[L']}\right) \hookrightarrow \pi B^q\left(X \times_k Y_{[L'']}\right)^{\mathrm{Gal}(L''/L')},$$

so

$$\pi B^q\left(X \times_k Y_{[L'']}\right)^{\mathrm{Gal}(L''/L')} = \mathrm{Hom}\left(Y_{[L']}^{\mathrm{prim}}, M\right) = \mathrm{Hom}\left(Y_{[L]}^{\mathrm{prim}}, M\right),$$

which means that $\alpha$ belongs to the image of $\mathrm{Hom}\left(Y_{[L]}, M\right)$, and thus, $\mathbb{B}^{[0]}(M)^{U_L} = \mathrm{Hom}\left(Y_{[L]}, M\right)$.

As $\mathbb{B}^{[0]}(M)^{U_L} = \mathrm{Hom}\left(Y_{[L]}, M\right) = \pi B^q(X \times_k Y_{[L]})$ and $\pi B^q(X_F)^{U_L} = \pi B^q(X_L)$, it suffices to show that $\pi B^q(X \times_k Y_{[L]}) = \pi B^q(X_L)$ for $\pi = \Delta_{k(X)}$ and any sufficiently big (with $\dim Y_{[L]} \geq q$) subfield $L$ finitely generated over $k$. We may assume that $F$ is algebraic over $L$, so $n < \infty$. Then, by Lemma 3.4, the natural map $Z^q(k(X) \otimes_k L) \longrightarrow B^q(X \times_k Y_{[L]})$ is surjective, and thus, the composition $Z^q(k(X) \otimes_k F) = \varinjlim_L Z^q(k(X) \otimes_k L) \longrightarrow \varinjlim_L \pi B^q(X \times_k Y_{[L]}) = \varinjlim_L \mathrm{Hom}\left(Y_{[L]}, M\right) = \mathbb{B}^{[0]}(M)$ is also surjective.

This implies that there exists a natural embedding of the $\mathbb{Q}$-algebra $\mathrm{End}_G \mathbb{B}^{[0]}(M)$ into the space $\mathrm{Hom}_G\left(Z^q(k(X) \otimes_k F), \mathbb{B}^{[0]}(M)\right)$.

Using Proposition 3.17 one gets

$$\begin{aligned}\mathrm{Hom}_G\left(Z^q(k(X) \otimes_k F), \mathbb{B}^{[0]}(M)\right) &= \mathrm{Hom}\left(Y_{[k(X)]}, M\right) \\ &= \mathrm{Hom}\left((Y_{[k(X)]}, \Delta_{k(X)}), M\right) = \pi B^q(X \times_k X)\Delta_{k(X)}.\end{aligned}$$

By Corollary 3.18, the latter coincides with $B^q(k(X) \otimes_k k(X))$, which is the same as $\mathrm{End}_G\left(\pi B^q(X_F)\right)$.

By [Jan] and Lemma 3.2, the $G$-module $\mathbb{B}^{[0]}(M)$ is semi-simple. As $\mathbb{B}^{[0]}(M)$ surjects onto $\pi B^q(X_F)$ and $\mathrm{End}_G \mathbb{B}^{[0]}(M) \subseteq \mathrm{End}_G \pi B^q(X_F)$, one has $\mathbb{B}^{[0]}(M) = \pi B^q(X_F)$. $\square$

**Corollary 3.20.** $\Delta_{k(X)} B^d(X \times_k Y) = B^d(k(X) \otimes_k k(Y))$ *and the group* $\Delta_{k(X)} B^q(X \times_k Y)$ *vanishes for any* $q < d := \dim X$*, any irreducible smooth proper* $k$*-variety* $X$ *and any irreducible smooth* $k$*-variety* $Y$*.*

*Proof.* $\Delta_{k(X)} B^d(X \times_k Y) = \mathrm{Hom}(Y, (X, \Delta_{k(X)}))$. By Proposition 3.19,

$$\mathrm{Hom}(Y, (X, \Delta_{k(X)})) = \Delta_{k(X)} B^d\left(X_{k(Y)}\right).$$

By Lemma 3.16, $\Delta_{k(X)} B^d\left(X_{k(Y)}\right) = B^d(k(X) \otimes_k k(Y))$.

Suppose that $q < d$. Consider the projection $X \times_k Y \times \mathbb{P}^{d-q} \xrightarrow{p} X \times_k Y$. The pull-back induces an embedding $\Delta_{k(X)} B^d(X \times_k Y) \xrightarrow{p^*} \Delta_{k(X)} B^d(X \times_k Y \times \mathbb{P}^{d-q})$, which is an isomorphism, since $\Delta_{k(X)} B^d(X \times_k Y)$ coincides with $B^d(k(X) \otimes_k k(Y))$, and $\Delta_{k(X)} B^d(X \times_k Y \times \mathbb{P}^{d-q}) = B^d(k(X) \otimes_k k(Y \times \mathbb{P}^{d-q}))$ coincides with $B^d(k(X) \otimes_k k(Y))$.

The push-forward induces a surjection $B^d(X \times_k Y \times \mathbb{P}^{d-q}) \xrightarrow{p_*} B^q(X \times_k Y)$. On the other hand, the composition $B^d(X \times_k Y) \xrightarrow{p_* p^*} B^q(X \times_k Y)$ is zero, so $\Delta_{k(X)} B^q(X \times_k Y) = 0$. □

*3.3 "Polarization" on $B^n(k(X) \otimes_k F)$ and polarizable $G$-modules*

**Proposition 3.21.** *For any irreducible $k$-variety $X$ of dimension $n$ there is a symmetric $G$-equivariant non-degenerate pairing*

$$B^n(k(X) \otimes_k F) \otimes B^n(k(X) \otimes_k F) \xrightarrow{\langle\ ,\ \rangle} \mathbb{Q}(\chi)$$

*such that $\langle p^*(\cdot),\ \cdot\ \rangle = \langle\ \cdot\ , p_*(\cdot)\rangle$ for any generically finite rational map $p$. In particular, $\langle\ ,\ \rangle$ induces non-degenerate pairings between the submodules $W := \pi B^n(k(X) \otimes_k F)$ and ${}^tW := {}^t\pi B^n(k(X) \otimes_k F)$ for all projectors $\pi \in B^n(k(X) \otimes_k k(X))$.*

*If for $(n-1)$-cycles on $2n$-dimensional complex varieties the numerical equivalence coincides with the homological one, then $\langle\ ,\ \rangle$ is $(-1)^n$-definite. In particular, this holds for $n \leq 2$.*

*Proof.* We may suppose that $X$ is a smooth projective variety over $k$. For a pair $\alpha, \gamma \in B^n(k(X) \otimes_k F)$ fixed by a compact open subgroup $U \subset G$ we define $\langle \alpha, \gamma\rangle \in \mathbb{Q}(\chi)$ by $\langle \widehat{\alpha} \cdot \widehat{\gamma}\rangle \cdot [U]$, where $\widehat{\alpha}, \widehat{\gamma}$ are the images of $\alpha, \gamma \in B^n(k(X) \otimes_k k(Y_U))$ in $B^n(X \times_k Y_U)$ in the sense of Proposition 3.14. Here $Y_U$ is a smooth proper variety over $k$ with the function field $k(Y_U)$ identified with $F^U$, and $\langle \cdot \rangle$ is the intersection form on $B^n(X \times_k Y_U)$. By the projection formula, $\langle \alpha, \gamma\rangle$ is independent of the choices, and $\langle p^*(\cdot),\ \cdot\ \rangle = \langle\ \cdot\ , p_*(\cdot)\rangle$.

For a triplet of smooth proper varieties $X_1, X_2, X_3$, a triplet of integers $a, b, c \geq 0$ with $a + b + c = \dim(X_1 \times X_2 \times X_3)$ and a triplet $\alpha \in A^a(X_1 \times X_2), \beta \in A^b(X_2 \times X_3), \gamma \in A^c(X_1 \times X_3)$ one has $\langle \alpha \circ \beta \cdot \gamma\rangle = \langle \beta \cdot {}^t\alpha \circ \gamma\rangle = \langle \alpha \cdot \gamma \circ {}^t\beta\rangle$. For any $\alpha \in W - \{0\}$ fixed by $U$ there is $\beta \in B^n(X \times_k Y_U)$ such that $\langle \widehat{\alpha} \cdot \beta\rangle \neq 0$. Then, as $\widehat{\alpha} = \pi \circ \widehat{\alpha} \circ \Delta_{k(Y_U)}$, one has $\langle \widehat{\alpha} \cdot \beta\rangle = \langle \widehat{\alpha} \cdot {}^t\pi \circ \beta \circ \Delta_{k(Y_U)}\rangle \neq 0$. As ${}^t\pi \circ \beta \circ \Delta_{k(Y_U)} \in {}^tW$, this shows that $\langle\ ,\ \rangle$ induces a non-degenerate pairing between $W$ and ${}^tW$ for an arbitrary projector $\pi$.

By Lemma 3.13, $\mathrm{Hom}_{\mathcal{H}(U)}(B^{n+1}(X \times_k Y_U), B^n(k(X) \otimes_k k(Y_U))) = 0$. By the semi-simplicity, this implies that the composition of the embedding of $\mathcal{H}(U)$-modules $B^n(k(X) \otimes_k k(Y_U)) \hookrightarrow B^n(X \times_k Y_U)$ with $B^n(X \times_k Y_U) \xrightarrow{\cdot [L]} B^{n+1}(X \times_k Y_U)$ is zero for any $L \in \mathrm{NS}(X)$. Interchanging the roles of $X$ and $Y_U$, we see that the image of $B^n(k(X) \otimes_k k(Y_U))$ in $B^n(X \times_k Y_U)$ is annihilated by any $L \in \mathrm{NS}(X) \bigoplus \mathrm{NS}(Y_U) \subseteq \mathrm{NS}(X \times_k Y_U)$, i.e., it consists of

primitive elements. Then, by the Hodge index theorem, if for $(n-1)$-cycles on $2n$-dimensional complex varieties the numerical equivalence coincides with the homological one, then the pairing $B^n(k(X)\otimes_k F)\otimes B^n(k(X)\otimes_k F)\xrightarrow{\langle\ ,\ \rangle}\mathbb{Q}(\chi)$ is $(-1)^n$-definite. $\square$

**Proposition 3.22.** *Let $V$ be a finite-dimensional $\mathbb{Q}$-vector space with a positive definite symmetric pairing $V\otimes V\xrightarrow{\langle\ ,\ \rangle}\mathbb{Q}$ and with an action $H\longrightarrow \mathrm{GL}(V)$, $p\longmapsto\sigma_p$ of a subgroup $H\subseteq\mathbb{Q}_+^\times$, generated by almost all primes, such that $\langle\sigma_p x,\sigma_p y\rangle = p\cdot\langle x,y\rangle$ for all $x,y\in V$ and for all $p\in H$. Then $V=0$.*

*Proof.* Since a $\overline{\mathbb{Q}}$-multiple of $\sigma_p$ is orthogonal, $\sigma_p$ is diagonizable over $\overline{\mathbb{Q}}$ for all $p\in H$. As the group $H$ is abelian, there is a basis $\{e_i\}$ of $V\otimes\overline{\mathbb{Q}}$ and characters $\lambda_i : H\longrightarrow\overline{\mathbb{Q}}^\times$ such that $\sigma_r e_i=\lambda_i(r)\cdot e_i$. Note, that the elements $e_i$ belong to $V\otimes K$ for a finite extension $K$ of $\mathbb{Q}$, so the characters factor as $\lambda_i : H\longrightarrow K^\times$.

For each embedding $\tau : K\hookrightarrow\mathbb{C}$ we define an hermitian form $\langle x,y\rangle_\tau := \langle\tau x,\overline{\tau y}\rangle$, where $\langle\ \ ,\ \ \rangle$ is the bilinear form on $V\otimes\mathbb{C}$ induced by $\langle\ \ ,\ \ \rangle$ and $\overline{\phantom{x}}$ is the complex conjugation on $V\otimes\mathbb{C}$. One has $0\neq\langle\sigma_p e_i,\sigma_p e_i\rangle_\tau = \tau(\lambda_i(p))\overline{\tau(\lambda_i(p))}\cdot\langle e_i,e_i\rangle_\tau = p\cdot\langle e_i,e_i\rangle_\tau$, so $|\tau(\lambda_i(p))| = p^{1/2}$ for any $\tau$.

Let $L$ be the subfield in the normalization of $K$ over $\mathbb{Q}$ generated by all the conjugates over $\mathbb{Q}$ of the image of $\lambda_1$, so $L$ is a Galois extension of $\mathbb{Q}$. For any embedding $\tau : K\hookrightarrow\mathbb{C}$ the image of $L$ in $\mathbb{C}$ is invariant under the complex conjugation, since $\overline{\tau\lambda_1(p)} = p\cdot(\tau\lambda_1(p))^{-1}$. As the latter does not depend on the embedding, the complex conjugation induces an element $c$ in the center of $\mathrm{Gal}(L/\mathbb{Q})$. $L$ cannot be totally real, since then it would contain elements $p^{1/2}$ for almost all prime $p$, and thus it would be of infinite degree. As the field of invariants $R$ of $c$ is totally real, $L = R(\sqrt{-\alpha})$ for some totally positive integer $\alpha\in\mathcal{O}_R$. Let $d=[R:\mathbb{Q}]$.

As the subgroup $\{1,c\}$ is normal in $\mathrm{Gal}(L/\mathbb{Q})$, the field $R$ is a Galois extension of $\mathbb{Q}$. By the Chebotarëv density theorem, there are infinitely many rational primes $p$ corresponding to the element (=the conjugacy class of) $c$. As restriction of $c$ to $R$ is trivial, such ideals $(p)$ split completely in the extension $R/\mathbb{Q}$, i.e., $(p)=\wp_1\cdots\wp_d$. On the other hand the ideals $\wp_1,\dots,\wp_d$ stay prime in the extension $L/R$ (one of them should stay prime, since $c$ is non-trivial, and the others lie in the $\mathrm{Gal}(L/\mathbb{Q})$-orbit of such one).

Now let integers $x,y\in\mathcal{O}_R$ be such that $x^2+\alpha y^2\in(p)\subseteq\wp_j$ for any $j$. Since $\wp_j$ remains prime in $\mathcal{O}_L$, one has $x+y\sqrt{-\alpha}\in\wp_j$ for a choice of $\sqrt{-\alpha}$. Since $\wp_j$ is $c$-invariant, $2x, 2y\sqrt{-\alpha}\in\wp_j$, and thus, $x,y\in\wp_j$ for $p$ big enough with respect to $\alpha$. So we get $x,y\in\wp_1\cdots\wp_d=(p)$.

If there is an element in $L$ with the modulus $p^{1/2}$ with respect to any complex embedding of $L$, then there are non-zero integers $x,y,z\in\mathcal{O}_R$ of minimal possible norm of $xyz$ such that $x^2+\alpha y^2=pz^2$. As this implies $z^2\in(p)$, one should also have $z\in(p)$ if $L$ is unramified over $p$, and therefore, the triplet $(x/p,y/p,z/p)$ also satisfies the above conditions and has a smaller "norm". This is contradiction. $\square$

We shall say that an admissible $G$-module $W$ is *polarizable* (when $n < \infty$) if there is a symmetric positive definite $G$-equivariant pairing $W \otimes W \longrightarrow \mathbb{Q}(\chi)$.

**Corollary 3.23.** *Let $W$ be a polarizable $G$-module. Let $L$ be an extension of $k$ in $F$. Then $W^{U_{L(x)}} = 0$ for any $x \in F$ transcendental over $L$.*

*Proof.* Since $W$ is smooth, there is a finitely generated extension $L_1$ of $k$ such that the stabilizer of an element $w \in W^{U_{L(x)}}$ contains the open subgroup $\langle U_{L_1}, U_{L(x)} \rangle$. By Lemma 2.1, the latter contains $U_{L_2(x) \cap L_3(x)} = U_{L_4(x)}$, where $L_3$ is generated over $L$ by a transcendence basis of $F$ over $L(x)$, $L_2$ is generated over $L_1$ by a transcendence basis of $L_3$ over $k$, and $L_4 := L_2(x) \bigcap L_3$ is finitely generated over $k$. We replace $L$ by $L_4$, thus assuming, that $L$ is finitely generated over $k$.

As $U_{L(x)} \subset U_{L(x^p)} = \sigma U_{L(x)} \sigma^{-1}$, where $\sigma x = x^p$ and $\sigma|_L = id$, the element $\sigma$ induces an isomorphism $W^{U_{L(x)}} \xrightarrow{\sim} W^{U_{L(x^p)}}$, and $W^{U_{L(x^p)}} \subseteq W^{U_{L(x)}}$, the dimension argument shows that $\sigma$ induces an automorphism of $W^{U_{L(x)}}$. For any $w \in W$ one has $\langle \sigma w, \sigma w \rangle = \chi(\sigma) \cdot \langle w, w \rangle$, so Proposition 3.22 says that $W^{U_{L(x)}} = 0$. $\square$

**Corollary 3.24.** *Any finite-dimensional polarizable $G$-module is zero.*

*Proof.* This follows from Corollary 2.10 and Corollary 3.23. $\square$

## 4 Morphisms between certain $G$-modules and matrix coefficients

### *4.1 Two remarks on the $G$-modules $F/k$ and $F^\times/k^\times$*

**Proposition 4.1.** *For any $1 \le n \le \infty$ the $G^\circ$-modules $F/k$ and $F^\times/k^\times$ are irreducible.*

*Proof.* Let $A$ be the additive subgroup of $F$ generated by the $G^\circ$-orbit of some $x \in F - k$. For any $y \in A - k$ one has $\frac{2}{y^2-1} = \frac{1}{y-1} - \frac{1}{y+1}$. As $\frac{1}{y-1}$ and $\frac{1}{y+1}$ are in the $G^\circ$-orbit of $y$, this implies that $y^2 \in A$. As for any $y, z \in A$ one has $yz = \frac{1}{4}((y+z)^2 - (y-z)^2)$, the group $A$ is a subring of $F$.

Let $M$ be the multiplicative subgroup of $F^\times$ generated by the $G^\circ$-orbit of some $x \in F - k$. Then for any $y, z \in M$ one has $y + z = z(y/z + 1)$, so if $y/z \notin k$ then $y + z \in M$, and thus, $M \bigcup \{0\}$ is a $G^\circ$-invariant subring of $F$.

Since the $G^\circ$-orbit of an element $x \in F - k$ contains all elements of $F - \overline{k(x)}$, if $n \ge 2$ then each element of $F$ is the sum of a pair of elements in the orbit. Any $G^\circ$-invariant subring in $F$, but not in $k$, is a $k$-subalgebra, so if $n = 1$ then $\mathrm{Gal}(F/\mathbb{Q}(G^\circ x)) \subset G^\circ$ is a compact subgroup normalized by $G^\circ$. Then by Theorem 2.9 we have $\mathrm{Gal}(F/\mathbb{Q}(G^\circ x)) = \{1\}$. As any element of $\mathbb{Q}(G^\circ x)$ is the fraction of a pair of elements in $\mathbb{Z}[G^\circ x]$ and for any $y \in F - k$ the element $1/y$ belongs to the $G^\circ$-orbit of $y$, one has $\mathbb{Z}[G^\circ x] = F$. $\square$

**Proposition 4.2.** *For any $1 \le n \le \infty$ the annihilator of $F/k$ in $\mathbf{D}_k$ and the annihilator of $F^\times/k^\times$ in $\mathbf{D}_{\mathbb{Q}}$ are trivial.*

*Proof.* Let $\sum_{j=1}^N a_j \cdot \sigma_j$ be the image of some $\alpha \in \mathbf{D}_E$ in $E[G/U_L]$ for an open subgroup $U_L$, an integer $N \geq 1$, some $a_j \in E$ and some pairwise distinct embeddings $\sigma_j : L \overset{/k}{\hookrightarrow} F$. Here $E$ is either $k$, or $\mathbb{Q}$. Let $\tau \in G$ be such an element that the embeddings $\sigma_1, \dots, \sigma_N, \tau\sigma_1, \dots, \tau\sigma_N$ are pairwise distinct.

Suppose $\alpha$ annihilates $F/k$. For any $x \in L$ one has $\sum_{j=1}^N a_j \cdot \sigma_j x \in k$, and thus, $\sum_{j=1}^N a_j \cdot \sigma_j x + \sum_{j=1}^N (-a_j) \cdot \tau\sigma_j x = 0$ for all $x \in L$. Then, by Artin's theorem on independence of characters, $a_1 = \dots = a_N = 0$.

Similarly, suppose that $\alpha$ annihilates $F^\times/k^\times$. We may suppose that $a_j \in \mathbb{Z}$. For any $x \in L^\times$ one has $\prod_{j=1}^N (\sigma_j x)^{a_j} \in k^\times$, and thus, $\prod_{j=1}^{2N} (\sigma_j x)^{a_j} = 1$ for all $x \in L^\times$, where $\sigma_j = \tau\sigma_{j-N}$ and $a_j = -a_{j-N}$ for $N < j \leq 2N$.

Let $\tau_1, \dots, \tau_M$ be a collection of pairwise distinct embeddings of $L$ into $F$ and $b_1, \dots, b_M$ be such a collection of non-zero integers that $\sum_{j=1}^M |b_j|$ is minimal among those for which $\prod_{j=1}^M (\tau_j x)^{b_j} = 1$ for all $x \in L^\times$. Then $\prod_{j=1}^M (\tau_j x + 1)^{b_j} - \prod_{j=1}^M (\tau_j x)^{b_j} = 0$, which is equivalent to

$$\prod_{j:b_j>0} (\tau_j x + 1)^{b_j} \prod_{j:b_j<0} (\tau_j x)^{-b_j} - \prod_{j:b_j>0} (\tau_j x)^{b_j} \prod_{j:b_j<0} (\tau_j x + 1)^{-b_j} = 0.$$

We rewrite this as $\sum_{c_1,\dots,c_M} A_{c_1,\dots,c_M} \prod_{j=1}^M (\tau_j x)^{c_j} = 0$, where $0 \leq c_j \leq |b_j|$ and $\sum_{j=1}^M c_j < \sum_{j=1}^M |b_j|$. Then $A_{c_1,\dots,c_M}$ are integers which are all non-zero if either $c_j = b_j$ for all $j$ with $b_j > 0$, or $c_j = -b_j$ for all $j$ with $b_j < 0$. By Artin's theorem on independence of characters, $\prod_{j=1}^M (\tau_j x)^{c_j} = \prod_{j=1}^M (\tau_j x)^{c'_j}$ for a pair of distinct collections $(c_j)$ and $(c'_j)$ as before, with $|c_j - c'_j| < |b_j|$ for some $j$, and for all $x \in L$. But then $\sum_{j=1}^M |c_j - c'_j| < \sum_{j=1}^M |b_j|$, contradicting our assumptions, so $a_1 = \dots = a_N = 0$. $\square$

### *4.2 Morphisms between certain $G$- and $G^\circ$-modules*

**Proposition 4.3.** *Let $\mathcal{A}$ and $\mathcal{B}$ be simple commutative group schemes over $k$. Then for any $1 \leq n \leq \infty$*

$$\mathrm{Hom}_{\text{group schemes}/k}(\mathcal{A}, \mathcal{B})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Hom}_G(W_{\mathcal{A}}, W_{\mathcal{B}}) \xrightarrow{\sim} \mathrm{Hom}_{G^\circ}(W_{\mathcal{A}}, W_{\mathcal{B}}).$$

*Proof.*

1. First, consider the cases $\mathcal{A}, \mathcal{B} \in \{\mathbb{G}_a, \mathbb{G}_m\}$. Let $\varphi \in \mathrm{Hom}_{G^\circ}(W_{\mathcal{A}}, W_{\mathcal{B}})$ and $x \in W_{\mathcal{A}} - \{0\}$. Then $\varphi(x)$ is fixed by the stabilizer $\mathrm{St}_x$ of $x$ in $G^\circ$. The group $\mathrm{St}_x$ fits into an exact sequence $1 \longrightarrow U_{k(x)} \bigcap G^\circ \longrightarrow \mathrm{St}_x \longrightarrow \mathcal{A}(k) \longrightarrow 1$. As $W_{\mathcal{B}}^{U_{k(x)} \bigcap G^\circ} = (\mathcal{B}(k(x))/\mathcal{B}(k))_{\mathbb{Q}}$ and $\mathcal{A}(k)$ acts on $k(x)$ by the affine linear substitutions of $x$, one has $W_{\mathcal{B}}^{\mathrm{St}_x} = (\mathcal{B}(k(x))/\mathcal{B}(k))_{\mathbb{Q}}^{\mathcal{A}(k)}$.
   - If $\mathcal{A} = \mathbb{G}_m$ and $\mathcal{B} = \mathbb{G}_a$ then $W_{\mathcal{B}}^{\mathrm{St}_x} = (k(x)/k)^{k^\times} = 0$, so $\varphi = 0$.
   - If $\mathcal{A} = \mathbb{G}_a$ and $\mathcal{B} = \mathbb{G}_m$ then $W_{\mathcal{B}}^{\mathrm{St}_x} = (k(x)^\times/k^\times)_{\mathbb{Q}}^k = 0$, so $\varphi = 0$.

– If $\mathcal{A} = \mathcal{B} = \mathbb{G}_m$ then $W_{\mathcal{B}}^{\mathrm{St}_x} = (k(x)^\times/k^\times)_{\mathbb{Q}}^{k^\times} = \{x^\lambda \mid \lambda \in \mathbb{Q}\}$, so $\varphi(x) = x^\lambda$ for some $\lambda \in \mathbb{Q}$. This implies that $\varphi(\sigma x) = (\sigma x)^\lambda$ for all $\sigma \in G^\circ$. As $F^\times/k^\times$ is an irreducible $G^\circ$-module, one has $\varphi y = y^\lambda$ for all $y \in F^\times/k^\times$.
– If $\mathcal{A} = \mathcal{B} = \mathbb{G}_a$ then $W_{\mathcal{B}}^{\mathrm{St}_x} = (k(x)/k)^k = \{\lambda \cdot x \mid \lambda \in k\}$, so $\varphi(x) = \lambda \cdot x$ for some $\lambda \in k$. This implies that $\varphi(\sigma x) = \lambda \cdot \sigma x$ for all $\sigma \in G^\circ$. As $F/k$ is an irreducible $G^\circ$-module, one has $\varphi y = \lambda \cdot y$ for all $y \in F/k$.

2. Fix a smooth irreducible curve $Z \subseteq \mathcal{A}$ over $k$. For any generic point $x : k(Z) \overset{/k}{\hookrightarrow} F$ and any $G^\circ$-homomorphism $W_{\mathcal{A}} \xrightarrow{\varphi} W_{\mathcal{B}}$ the element $\varphi(x)$ belongs to the space $W_{\mathcal{B}}^{U_{x(k(Z))}\cap G^\circ} = (\mathcal{B}(x(k(Z)))/\mathcal{B}(k))_{\mathbb{Q}}$, and therefore, there is an integer $N_x \geq 1$ and a rational map $Z \overset{h_x}{\dashrightarrow} \mathcal{B}$ defined over $k$ such that $\varphi(x) = \frac{1}{N_x} h_x(x)$. This implies that if $\varphi \neq 0$ and either $\mathcal{A} = \mathbb{G}_m$ or $\mathcal{A} = \mathbb{G}_a$ then $\mathcal{B} = \mathbb{G}_m$ or $\mathcal{B} = \mathbb{G}_a$.
3. Now suppose that $\mathcal{A}$ and $\mathcal{B}$ are simple abelian varieties over $k$. Then the maps $h_x$ are regular and factor as $Z \longrightarrow J_Z \xrightarrow{\varphi_x} \mathcal{B}$ for some homomorphisms $\varphi_x$ from the Jacobian $J_Z$.
4. If $\mathcal{A}$ and $\mathcal{B}$ are not isogeneous, there is such a curve $Z$ that any homomorphism of the group schemes $J_Z \longrightarrow \mathcal{B}$ is trivial, and thus $\varphi(x) = 0$, so we may suppose that $\mathcal{A} = \mathcal{B}$.
5. There is such a curve $Z$ that
$$\mathrm{End}_{\mathcal{A}} := \mathrm{End}_{\text{group scheme}/k}(\mathcal{A})_{\mathbb{Q}} \xrightarrow{\sim} \mathrm{Hom}_{\text{group schemes}/k}(J_Z, \mathcal{A})_{\mathbb{Q}}.$$
Then one can consider $\varphi_x$ as an element of $\mathrm{End}_{\mathcal{A}}$, i.e., $\varphi(x) = \frac{1}{N_x}\varphi_x(x)$.
6. For any pair of generic points $x_1, x_2 \in Z(F)$ there are generic points
$$t, y_1, \ldots, y_M, z_1, \ldots, z_{M'}$$
of $Z(F)$ such that the elements of both collections $(x_1, t, y_1, \ldots, y_M)$ and $(x_2, t, z_1, \ldots, z_{M'})$ are linearly independent in $W_{\mathcal{A}}$ over the algebra $\mathrm{End}_{\mathcal{A}}$, and $u := x_1 + t + \sum_j y_j$ and $v := x_2 + t + \sum_j z_j$ are generic points of $Z(F)$.
Then, by definition, $\varphi(u)$ coincides with
$$\frac{1}{N_u}\varphi_u(u) = \frac{1}{N_u}\left(\varphi_u(x_1) + \varphi_u(t) + \sum_j \varphi_u(y_j)\right),$$
and, as $\varphi$ is a homomorphism, $\varphi = \frac{1}{N_{x_1}}\varphi_{x_1}(x_1) + \frac{1}{N_t}\varphi_t(t) + \frac{1}{N_{y_1}}\varphi_{y_1}(y_1) + \cdots + \frac{1}{N_{y_M}}\varphi_{y_M}(y_M)$. We can assume that $N_u = N_v = N_{x_1} = N_{x_2} = N_t$. Then one has $\varphi_u = \varphi_{x_1} = \varphi_t$. Similarly, one has $\varphi_v = \varphi_{x_2} = \varphi_t$, and therefore, the restriction of $\varphi$ to the set of generic points of $Z$ coincides with the restriction of $\psi$ to the set of generic points of $Z$, for some $\psi \in \mathrm{End}_{\mathcal{A}}$.
As generic points of $Z$ generate $\mathcal{A}(F)$ as an abstract group, this implies that $\varphi = \psi$, i.e., that $\varphi \in \mathrm{End}_{\mathcal{A}}$.

7. There remain the cases where $\mathcal{A}$ is an abelian variety and $\mathcal{B}$ is either $\mathbb{G}_a$ or $\mathbb{G}_m$. As $W_{\mathcal{B}}$ is an irreducible $G^\circ$-module, any non-zero $G^\circ$-homomorphism to $W_{\mathcal{B}}$ is surjective. Any surjection of smooth representations of $G^\circ$ induces a surjection of their subspaces fixed by a compact subgroup $K$ of $G$. Taking $K = U_{L'}$, where $L'$ is a purely transcendental extension of $k$, we get contradiction showing that $\mathrm{Hom}_{G^\circ}(W_{\mathcal{A}}, W_{\mathcal{B}}) = 0$. □

**Corollary 4.4.** *For any pair of pure* 1*-motives $M_1, M_2$ one has*

$$\mathrm{Hom}_G(\mathfrak{B}^1(M_1), \mathfrak{B}^1(M_2)) = \mathrm{Hom}_{G^\circ}(\mathfrak{B}^1(M_1), \mathfrak{B}^1(M_2)). \qquad \square$$

**Corollary 4.5.** *For any $G$-module $W$ there is at most* 1 *character $\psi$ such that $W(\psi) \cong \mathfrak{B}^1(M)$ for a pure* 1*-motive $M$.*

*Proof.* Suppose that $W$ is irreducible, $W(\psi_1) \cong \mathfrak{B}^1(M_1)$, $W(\psi_2) \cong \mathfrak{B}^1(M_2)$ for some pure 1-motives $M_1$ and $M_2$ and some characters $\psi_1 \neq \psi_2$. By the fully faithfulness and the previous corollary,

$$\mathrm{Hom}(M_1, M_2) = \mathrm{Hom}_G(\mathfrak{B}^1(M_1), \mathfrak{B}^1(M_2)) = \mathrm{Hom}_{G^\circ}(\mathfrak{B}^1(M_1), \mathfrak{B}^1(M_2)),$$

so $\mathrm{Hom}(M_1, M_2) = \mathrm{Hom}_{G^\circ}(W(\psi_1), W(\psi_2))$, which non-zero, since the $G^\circ$-modules $W(\psi_1)$ and $W(\psi_2)$ are isomorphic, and thus, $M_1$ and $M_2$ are isomorphic, which implies that $W(\psi_1) \cong W(\psi_2)$ as $G$-modules. We may assume that $\psi_2 = 1$. Set $\psi = \psi_1 \neq 1$.

The $G$-module $W(\psi)$ coincides with $W$ as a vector space, but $G$ acts by $(\sigma, w) \longmapsto \psi(\sigma) \cdot \sigma w$. Suppose that there is an isomorphism $W \longrightarrow W(\psi)$, i.e., an automorphism $W \xrightarrow{\lambda} W$ such that $\lambda(\sigma w) = \psi(\sigma) \cdot \sigma\lambda(w)$. Then $\lambda$ can be considered as an element of $\mathrm{End}_{\ker\psi}(W)$. As $\ker\psi$ contains $G^\circ$, the automorphism $\lambda$ can also be considered as an element of $\mathrm{End}_{G^\circ}(W)$.

This implies that $\mathrm{End}_{G^\circ}(W) \neq \mathrm{End}_G(W)$, contradicting the previous corollary.

If $W$ is not irreducible, it should be semi-simple anyway, so its irreducible summands are motivic, implying the Corollary. □

### *4.3 Non-compactness of supports of matrix coefficients*

A *matrix coefficient* of a smooth representation $W$ of a topological group is a function on the group of type $\langle \sigma w, \widetilde{w} \rangle$ for a vector $w \in W$ and a vector $\widetilde{w}$ in the dual representation with open stabilizer.

**Proposition 4.6.** *Suppose that $n < \infty$, a subgroup $H$ of $G$ contains $G^\circ$ and the supports of the matrix coefficients of a representation of $H$ are compact. Then this representation is zero.*

*Proof.* Let $\mathbb{C}$ be an algebraically closed extension of $k$ of cardinality strictly greater than the cardinality of $k$. Let $\omega \neq 0$ be an irreducible representation of $H$ over $\mathbb{C}$ with compact supports of the matrix coefficients. We may

replace $H$ by $G^\circ$, and replace $\omega$ by an irreducible subquotient of $\omega|_{G^\circ}$. Let $N$ be a compact open subgroup in $G^\circ$ such that $\omega(h_N) \neq 0$.

Along the same lines as, e.g., in Claim 2.11 of [BZ], one proves the Schur's lemma: the endomorphisms of a smooth irreducible $\mathbb{C}$-representation of $G^\circ$ are scalar. This allows one to modify Theorem 2.42 a) of [BZ] as follows. *For each open compact subgroup $N$ in $G^\circ$ there is an element $\varepsilon_N^\omega \in \mathbf{D}_{\mathbb{C}}$ such that $\omega(\varepsilon_N^\omega) = \omega(h_N)$ and $\pi(\varepsilon_N^\omega) = 0$ for any smooth irreducible $\mathbb{C}$-representation $\pi$ of $G^\circ$ distinct from $\omega$.*

As the $G^\circ$-module $F^\times/k^\times$ is irreducible, and $\mathrm{End}_{G^\circ}(F^\times/k^\times) = \mathbb{Q}$, the $\mathbb{C}$-representation $\pi = (F^\times/k^\times) \otimes \mathbb{C}$ of $G^\circ$ is irreducible.[10] On the other hand, the support of the matrix coefficient $\langle \sigma x, \widetilde{w} \rangle$ is not compact for any $x \in F^\times/k^\times - \{1\}$ and any vector $\widetilde{w} \neq 0$ in the representation dual to $\pi$, since the stabilizer of $x$ is not compact. This implies that there is an element $\varepsilon \in \mathbf{D}_{\mathbb{C}}$ such that $\omega(\varepsilon) = \omega(h_N)$ and $\pi(\varepsilon) = 0$, contradicting Proposition 4.2. $\square$

## 5 Some examples of (co-)homological calculations

### *5.1 Examples of* Ext*-calculation and torsors*

Let $\mathcal{S}m_G$ be the category of smooth $G$-modules. It is a full abelian subcategory in the category of $G$-modules.

**Proposition 5.1.** *Let $n = \infty$ and $\mathcal{A}$ be an irreducible commutative algebraic group over $k$.*

*Then* $\mathrm{Ext}^1_{\mathcal{S}m_G}(\mathcal{A}(F)_{\mathbb{Q}}, \mathbb{Q}) = \mathrm{Ext}^1_{G,\mathrm{cont}}(\mathcal{A}(F)_{\mathbb{Q}}, \mathbb{Q}) = 0$.

*Proof.* Let $0 \longrightarrow \mathbb{Q} \longrightarrow E \longrightarrow \mathcal{A}(F)_{\mathbb{Q}} \longrightarrow 0$ be a continuous extension, i.e., with closed stabilizers. A choice of a linear section $\mathcal{A}(F)_{\mathbb{Q}} \overset{s}{\hookrightarrow} E$ defines a splitting $E \cong \mathbb{Q} \bigoplus \mathcal{A}(F)_{\mathbb{Q}}$ as a $\mathbb{Q}$-vector space. The $G$-action is given by $\sigma(b, x) = (b + a_\sigma(x), \sigma x)$, where $a_\sigma(x + y) = a_\sigma(x) + a_\sigma(y)$ and $a_\sigma(\tau x) + a_\tau(x) = a_{\sigma\tau}(x)$. $E$ is continuous if and only if the subgroup $\{\sigma \in \mathrm{Stab}_x \mid a_\sigma(x) = 0\}$ is closed for any $x \in \mathcal{A}(F)_{\mathbb{Q}}$. In particular, the map $\mathrm{Stab}_x \overset{a(x)}{\longrightarrow} \mathbb{Q}$ given by $\sigma \longmapsto a_\sigma(x)$ is a homomorphism with closed kernel. As $a(x)$ is continuous, the image of any compact subgroup of $U_{k(x)}$ is a compact subgroup in $\mathbb{Q}$, i.e., 0. By Lemma 2.8, the subgroup generated by compact subgroups is dense in $U_{k(x)}$, so $U_{k(x)}$ is in the kernel of $a(x)$, and thus, $a(x)$ factors through $\mathrm{Stab}_x/U_{k(x)} \subseteq \mathcal{A}(k)_{\mathrm{tors}}$. Since any homomorphism from any torsion group to $\mathbb{Q}$ is zero, we get $a_\sigma(x) = 0$ for any $\sigma \in \mathrm{Stab}_x$, and therefore, $\mathrm{Stab}_{(b,y)} = \mathrm{Stab}_y$ for any $(b, y) \in E$. This implies also that for any $y \in \mathcal{A}(F)_{\mathbb{Q}}$, any $\tau \in G$ and any $\sigma \in \mathrm{Stab}_y$ one has $a_{\tau\sigma}(y) = a_\tau(y)$. In particular, if $x$ is a generic point of $\mathcal{A}$ and $H = \{\tau \in G \mid \tau x = \mu \cdot x \text{ for some } \mu \in \mathbb{Q}^\times\}$, then $a(x) : H \longrightarrow \mathbb{Q}$ factors

[10] Variant: $F/k$ is an irreducible $G^\circ$-module, $\mathbf{D}_{\mathbb{Q}} \hookrightarrow \mathrm{End}_k(F/k)$, $\mathrm{End}_{G^\circ}(F/k) = k$, and therefore, $(F/k) \otimes_k \mathbb{C}$ is an irreducible $\mathbb{C}[G^\circ]$-module.

through $\mathbb{Q}^\times \longrightarrow \mathbb{Q}$ and $p \cdot a_\tau(x) + a_\sigma(x) = a_{\tau\sigma}(x)$ for any $\sigma, \tau \in G$ such that $\sigma x = p \cdot x$ and $\tau x = q \cdot x$ for some $p, q \in \mathbb{Q}^\times$. Then $a_\sigma(x) = (p-1) \cdot c(x)$ for some $c(x) \in \mathbb{Q}$. Clearly, $c(m \cdot x) = m \cdot c(x)$ for any $m \in \mathbb{Q}^\times$.

Note, that $c(\ )$ is linear on the set of generic points of $\mathcal{A}$. Indeed, if $x$ and $y$ (considered as embeddings $x, y : k(\mathcal{A}) \overset{/k}{\hookrightarrow} F$) are algebraically independent over $k$ (i.e., $k(x,y) := x(k(\mathcal{A}))y(k(\mathcal{A})) \subset F$ is of transcendence degree $2 \dim \mathcal{A}$ over $k$) then there is an element $\sigma \in G$ such that $\sigma x = 2 \cdot x$ and $\sigma y = 2 \cdot y$, so $a_\sigma(x) = c(x)$, $a_\sigma(y) = c(y)$, and $a_\sigma(x+y) = c(x+y)$, so, by additivity of $a_\sigma$, one has $c(x+y) = c(x) + c(y)$. In general, for any collection of generic points $x_1, \dots, x_N$ there is some $z \in \mathcal{A}(F)$ such that the subfield $k(z, x_1, \dots, x_N)$ of $F$ is of transcendence degree $\dim \mathcal{A}$ over $k(x_1, \dots, x_N)$. By induction on $N$, one has $\sum_{j=1}^N m_j \cdot c(x_j) = c(z + \sum_{j=1}^N m_j \cdot x_j) - c(z)$. In particular, if $\sum_{j=1}^N m_j \cdot x_j = 0$ this means that $\sum_{j=1}^N m_j \cdot c(x_j) = 0$. This implies that $c(\ )$ extends to a linear functional on $\mathcal{A}(F)_\mathbb{Q}$.

Subtracting the coboundary of $c(\ )$, we may assume that $a_\sigma(y) = 0$ for any generic point $y \in \mathcal{A}(F)$ and any $\sigma \in G$ such that $\sigma y = \mu \cdot y$ for some $\mu \in \mathbb{Q}^\times$.

Fix a generic point $y \in \mathcal{A}(F)$. Any $G$-equivariant section over the subset of generic points of $\mathcal{A}$ is of type $\sigma y \overset{\alpha_{y,b}}{\longmapsto} \sigma(b, y)$ for some $b \in \mathbb{Q}$. As $G$ acts transitively on the set of generic points of $\mathcal{A}$, and the stabilizers of vectors in $E$ coincide with the stabilizers of their projections to $\mathcal{A}(F)_\mathbb{Q}$, this section is well-defined. For any $\mu \in \mathbb{Q}^\times$ let $\tau_\mu \in G$ be such an element that $\tau_\mu y = \mu \cdot y$. Then $\alpha_{y,b}(\mu \cdot \sigma y) = \alpha_{y,b}(\sigma\tau_\mu y) = \sigma\tau_\mu \alpha_{y,b}(y) = \sigma\tau_\mu(b, y) = (b + a_{\sigma\tau_\mu}(y), \sigma\tau_\mu y) = (b + a_\sigma(\tau_\mu y), \sigma\tau_\mu y) = \mu \cdot (b/\mu + a_\sigma(y), \sigma y)$, since $a_{\tau_\mu}(y) = 0$, so $\alpha_{y,b}(\mu \cdot x) = \mu \cdot \alpha_{y,b}(x)$ for any generic point $x \in \mathcal{A}(F)$ and any $\mu \in \mathbb{Q}^\times$ if and only if $b = 0$, i.e., $\alpha_{y,0}$ is the unique $G$-equivariant homogeneous (but, a priori, non-linear) section over the subset of generic points of $\mathcal{A}$. As $\alpha_{\sigma y,0}$ is also a $G$-equivariant homogeneous section over the subset of generic points of $\mathcal{A}$ for arbitrary $\sigma \in G$, one has $\alpha_{y,0}(\sigma y) = (a_\sigma(y), \sigma y) = \alpha_{\sigma y,0}(\sigma y) = (0, \sigma y)$, so $a_\sigma(y) = 0$ for any $\sigma \in G$ and any generic point $y$. Since any element of $\mathcal{A}(F)_\mathbb{Q}$ is a sum of generic points of $\mathcal{A}$, we get $a_\sigma(z) = 0$ for any $\sigma \in G$ and any $z \in \mathcal{A}(F)_\mathbb{Q}$. $\Box$

**Corollary 5.2.** *Let $n = \infty$ and $\mathcal{A}$ be an irreducible commutative algebraic group over $k$. Then one has*

$$\mathrm{Ext}^1_{\mathcal{S}m_G}(W_\mathcal{A}, \mathbb{Q}) = \mathrm{Ext}^1_{G,\mathrm{cont}}(W_\mathcal{A}, \mathbb{Q}) = \mathrm{Hom}(\mathcal{A}(k), \mathbb{Q}).$$

*Proof.* The functor $\mathrm{RHom}(-, \mathbb{Q})$ applied to $0 \longrightarrow \mathcal{A}(k)_\mathbb{Q} \longrightarrow \mathcal{A}(F)_\mathbb{Q} \longrightarrow W_\mathcal{A} \longrightarrow 0$ gives an exact sequence $\mathrm{Hom}(\mathcal{A}(F)_\mathbb{Q}, \mathbb{Q}) \longrightarrow \mathrm{Hom}(\mathcal{A}(k)_\mathbb{Q}, \mathbb{Q}) \longrightarrow \mathrm{Ext}^1(W_\mathcal{A}, \mathbb{Q}) \longrightarrow \mathrm{Ext}^1(\mathcal{A}(F)_\mathbb{Q}, \mathbb{Q})$, where the exterior groups are zero by Proposition 5.1. $\Box$

**Lemma 5.3.** *Let $n = \infty$, and either $\mathcal{A} = \mathbb{G}_m$, or $\mathcal{A} = \mathbb{G}_a$.*

*Then* $\mathrm{Ext}^1_{\mathcal{S}m_G}(\mathbb{Q}, W_\mathcal{A}) = 0$.

*Proof.* Let $0 \longrightarrow W_{\mathcal{A}} \longrightarrow E \longrightarrow \mathbb{Q} \longrightarrow 0$ be an exact sequence in $\mathcal{S}m_G$. The stabilizer of an element of $E$ projecting to $1 \in \mathbb{Q}$ contains an open subgroup $U_L$, so $E$ is a sum of $W_{\mathcal{A}}$ and a quotient of $\mathbb{Q}[G/U_L]$ by a submodule in the group $\mathbb{Q}[G/U_L]^{\circ}$ of degree-zero 0-cycles.

When restricted to $\mathbb{Q}[G/U_L]^{\circ}$, the projection $\mathbb{Q}[G/U_L] \longrightarrow E$ factors through $W_{\mathcal{A}}$. Denote it by $\alpha$. Fix elements $\sigma, \tau \in G$ such that $L$, $\sigma(L)$ and $\tau\sigma(L)$ are in general position, $\sigma|_L = \tau|_L$ and $\sigma^2|_L = id$. Then the generator $[1] - [\sigma]$ of $\mathbb{Q}[G/U_L]^{\circ}$ is fixed by $U_{L\sigma(L)}$, and therefore,

$$\alpha([1] - [\sigma]) = f(x, y) \in (\mathcal{A}(L\sigma(L))/\mathcal{A}(k))_{\mathbb{Q}}\,,$$
$$\alpha([\sigma] - [\tau\sigma]) = f(y, z) \in (\mathcal{A}(\sigma(L)\tau\sigma(L))/\mathcal{A}(k))_{\mathbb{Q}}\,,$$
$$\alpha([1] - [\tau\sigma]) = f(x, z) \in (\mathcal{A}(L\tau\sigma(L))/\mathcal{A}(k))_{\mathbb{Q}}\,,$$

where $f(-,-)$ is a rational function and $x, y, z$ denote collections of elements in $L, \sigma(L), \tau\sigma(L)$, respectively. Then $f(x, y)+f(y, z) = f(x, z)$. Taking $\frac{\partial^2}{\partial x \partial z}$ in the case $\mathcal{A} = \mathbb{G}_a$, or $\frac{\partial}{\partial x}\log \cdot \frac{\partial}{\partial z}\log$ in the case $\mathcal{A} = \mathbb{G}_m$, we get $f(x, z) = f(x) + g(z)$. As $\sigma([1] - [\sigma]) = -([1] - [\sigma])$, one has $f(x, y) = -f(y, x)$, and thus, $f(x, y) = f(x) - f(y)$ for some $f(x) \in L$. Let $\beta \in G$ be such an element that $\beta f(x) = 2f(x)$ (in the sense of the group law of $\mathcal{A}$). Then the image of $2 \cdot [1] - [\beta]$ in $E$ is fixed by $G$ and projects to $1 \in \mathbb{Q}$, so $E \cong \mathbb{Q} \oplus W_{\mathcal{A}}$. □

It will follow from Propositions 6.4 and 6.15 that $\mathrm{Ext}^1_{\mathcal{S}m_G}(\mathbb{Q}, W) = 0$ for any admissible representation $W$ of $G$ in the case $n = \infty$. (More generally, if $n = \infty$, $W_1, W_2 \in \mathcal{I}^q_G$ and $W_1$ is projective in $\mathcal{I}^q_G$ then, using Lemma 6.6, we get $\mathrm{Ext}^1_{\mathcal{S}m_G}(W_1, W_2) = \mathrm{Ext}^1_{\mathcal{S}m_G}(W_1, W_2) = \mathrm{Ext}^1_{\mathcal{I}^q_G}(W_1, W_2) = 0$. Now take $q = 0$, $W_1 = \mathbb{Q}$.)

In the next example, we wish to show that $\mathrm{Ext}^1_{\mathcal{S}m_G}(W_{\mathcal{A}}, W_{\mathbb{G}_m}) \neq 0$ for any abelian variety $\mathcal{A}$ over $k$. The $G$-module $\mathrm{Div}^{\circ}_{\mathbb{Q}} = \varinjlim_U \mathrm{Div}_{\mathrm{alg}}(Y_U)_{\mathbb{Q}}$, introduced before Proposition 3.11, fits into the exact sequence $0 \longrightarrow F^{\times}/k^{\times} \longrightarrow \mathrm{Div}^{\circ}_{\mathbb{Q}} \longrightarrow \mathrm{Pic}^{\circ}_{\mathbb{Q}} \longrightarrow 0$. By Proposition 3.11, any non-zero element of $\mathcal{A}^{\vee}(k)_{\mathbb{Q}}$ determines an embedding of $W_{\mathcal{A}}$ into $\mathrm{Pic}^{\circ}_{\mathbb{Q}}$, thus inducing an extension of $W_{\mathcal{A}}$ by $W_{\mathbb{G}_m}$ inside $\mathrm{Div}^{\circ}_{\mathbb{Q}}$. This extension is non-split, since any generic $F$-point $x$ of $\mathcal{A}$, considered as an element of $\mathcal{A}(F)_{\mathbb{Q}}$, identifies the space $\mathrm{Hom}_G(\mathcal{A}(F)_{\mathbb{Q}}, \mathrm{Div}^{\circ}_{\mathbb{Q}})$ with a subspace in $(\mathrm{Div}^{\circ}_{\mathbb{Q}})^{\mathrm{Stab}_x}$ which is the same as

$$\mathrm{Div}_{\mathrm{alg}}(\mathcal{A})_{\mathbb{Q}}^{\langle \text{translations by torsion elements in } \mathcal{A}(k)\rangle} = 0.$$

For a smooth $G$-group $\mathcal{A}$ denote by $H^1_{\mathcal{S}m}(G, \mathcal{A})$ the set of isomorphism classes of smooth $G$-torsors under $\mathcal{A}$, i.e., the set of classes of those $G$-torsors in $H^1(G, \mathcal{A})$ that become trivial on an open subgroup of $G$.

In particular, $H^1(G, \mathrm{GL}_r F)$ is the set of isomorphism classes of semi-linear $r$-dimensional representations of $G$ over $F$, and $H^1_{\mathcal{S}m}(G, \mathrm{GL}_r F)$ is its subset of smooth ones.

**Proposition 5.4.** *If $n = \infty$ then $H^1_{\mathcal{S}m}(G, \mathcal{A}(F)) = \{*\}$ for any algebraic group $\mathcal{A}$ over $k$.*

*Proof.* For any 1-cocycle $(a_\sigma)$ presenting a class in $H^1_{\mathcal{S}m}(G, \mathcal{A}(F))$ there is an extension $L$ of $k$ in $F$ of finite type such that $a_\xi = 1$ for any $\xi \in U_L$, so, as $a_{\sigma\tau} = a_\sigma \cdot \sigma a_\tau$, $a$ is a function on $G/U_L$, and $a_\sigma \in \mathcal{A}(L\sigma(L))$ for any $\sigma \in G$.

For any $\tau : L \overset{/k}{\hookrightarrow} F$ in general position with respect to $L$ there is some $\sigma : L \overset{/k}{\hookrightarrow} F$ such that $\sigma(L)$ and $\sigma\tau(L)$ are in general position with respect to $L$. Set $F' = \sigma(L)\tau(L)\sigma\tau(L) \subset F$. There is an $F'$-subalgebra $R$ of finite type in $LF' \subset F$ with the fraction field $LF'$ such that the fraction field of $R \bigcap L$ coincides with $L$ and $a_\sigma, a_\tau, a_{\sigma\tau} \in \mathcal{A}(R)$. There is a ring homomorphism $R \xrightarrow{s} F'$ identical on $F'$ and inducing a homomorphism $R \bigcap L \longrightarrow k$, so $s(a_\xi) \in \mathcal{A}(\xi(L))$ if $\xi \in \{\sigma, \tau, \sigma\tau\}$.

Clearly, $\sigma a_\tau \in \mathcal{A}(F')$, so $s(a_{\sigma\tau}) = s(a_\sigma \cdot \sigma a_\tau) = s(a_\sigma) \cdot \sigma a_\tau$, and thus, $a_\tau = \sigma^{-1} s(a_\sigma)^{-1} \cdot \sigma^{-1} s(a_{\sigma\tau})$. This implies that $a_\tau = f_\sigma^{-1} \cdot \tau f_{\sigma\tau}$, where $f_\xi := \xi^{-1} s(a_\xi) \in \mathcal{A}(L)$ and $\xi \in \{\sigma, \sigma\tau\}$.

In other words, for any $\tau : L \overset{/k}{\hookrightarrow} F$ in general position with respect to $L$ there exist $g_\tau, h_\tau \in \mathcal{A}(L)$ such that $a_\tau = g_\tau \cdot \tau h_\tau$. Any $\sigma : L \overset{/k}{\hookrightarrow} F$ in general position with respect to $L$ can be extended to $L\tau(L) \overset{/k}{\hookrightarrow} F$ in such a way that $L, \sigma(L)$ and $\sigma\tau(L)$ will be in general position, so $g_\sigma^{-1} \cdot a_{\sigma\tau} \cdot \sigma\tau h_\tau^{-1} = \sigma h_\sigma \cdot \sigma g_\tau$ is an element of $\mathcal{A}(L\sigma\tau(L))$ and simultaneously of $\mathcal{A}(\sigma(L))$, i.e., $h_\sigma \cdot g_\tau \in \mathcal{A}(k)$ for any $\sigma, \tau : L \overset{/k}{\hookrightarrow} F$ in general position with respect to $L$. This means that $h_\sigma = b_\sigma \cdot f$ and $g_\tau = f^{-1} \cdot c_\tau$, where $b_\sigma, c_\tau \in \mathcal{A}(k)$ and $f \in \mathcal{A}(L)$, so $a_\tau = f^{-1} \cdot c_\tau b_\tau \cdot \tau f$.

For any $\xi \in G$ there exist $\sigma$ and $\tau$ in general position with respect to $L$ such that $\xi = \sigma\tau$, and therefore, $a_\xi = (f^{-1} \cdot c_\sigma b_\sigma \cdot \sigma f) \cdot \sigma(f^{-1} \cdot c_\tau b_\tau \cdot \tau f) = f^{-1} \cdot (c_\sigma b_\sigma c_\tau b_\tau) \cdot \sigma\tau f = f^{-1} \cdot d_\xi \cdot \xi f$ for some $d_\xi \in \mathcal{A}(k)$, which means that the 1-cocycle $(a_\xi)$ is cohomological to the image in $H^1_{\mathcal{S}m}(G, \mathcal{A}(F))$ of an element of $H^1_{\mathcal{S}m}(G, \mathcal{A}(k)) = \mathrm{Hom}_{\mathcal{S}m}(G, \mathcal{A}(k))$, trivial by Theorem 2.9. $\square$

Remarks. 1. If $n < \infty$ then there exist uncountably many semi-linear smooth representations of $G$ over $F$ of any finite dimension. Namely, any extension of coefficients from $\mathbb{Q}$ to $F$ of a non-trivial finite-dimensional smooth $\mathbb{Q}$-representations of $G$ gives such semi-linear representations of $G$. (In fact, the natural map $H^1(G, \mathcal{A}(k)) \longrightarrow H^1(G, \mathcal{A}(F))$ is injective for any algebraic group $\mathcal{A}$ over $k$. Otherwise, for a pair of 1-cocycles $(a_\sigma)$ and $(a'_\sigma)$ on $G$ with values in $\mathcal{A}(k)$ there would exist such $B \in \mathcal{A}(F)$ that $\sigma B = a_\sigma \cdot B \cdot {a'_\sigma}^{-1}$. If $f(B) \notin k$ for some $f \in k(\mathcal{A})$ defined at $B$ then there is $\sigma \in G$ sending $f(B)$ out of the field of definition of $B$, so $\sigma B \neq A \cdot B \cdot A'$ for any $A, A' \in \mathcal{A}(k)$. $\square$)

2. If $n < \infty$ then there exist non-trivial finite-dimensional semi-linear smooth $F$-representations of $G^\circ$, e.g, $\Omega^q_{F/k}$ for any $1 \leq q \leq n$. (Moreover,

the class of $\det_F \Omega^q_{F/k} = \left(\Omega^n_{F/k}\right)^{\otimes_F \binom{n-1}{q-1}}$ in $H^1(G^\circ, \Omega^1_{F/k})$ under the $d\log$ map is non-trivial. Otherwise, for a generator $\omega = dx_1 \wedge \cdots \wedge dx_n$ of $\Omega^n_{F/k}$ there would exist some $\psi \in \Omega^1_{F/k}$ such that $\frac{d(\sigma\omega/\omega)}{\sigma\omega/\omega} = \sigma\psi - \psi$ for any $\sigma \in G^\circ$. For any $\sigma \in G^\circ$ with $\sigma x = Ax + B$ for an invertible $(n \times n)$-matrix $A$ over $k$ and a $k$-vector $B$ one has $\frac{d(\sigma\omega/\omega)}{\sigma\omega/\omega} = 0$, and thus, $\sigma\psi - \psi = 0$. But it is clear, that there are no non-zero 1-forms invariant under all such $\sigma$'s, so $\psi = 0$, and thus, $\tau\omega/\omega \in k$ for any $\tau \in G^\circ$. However, it is not the case if $\tau x_j = -x_j^{-1}$ for all $1 \le j \le n$. $\square$)

### *5.2 An example of $H_0$-calculation*

**Lemma 5.5.** *Let $n = \infty$, $X$ a smooth projective variety over $k$, $\mathcal{A}$ an irreducible commutative algebraic group over $k$, and either $W = CH_0(X_F)^0_{\mathbb{Q}}$, or $W = \mathcal{A}(F)_{\mathbb{Q}}$. Then for any open subgroup $U_L$ in $G$ one has $H_0(U_L, W) = 0$.*

*Proof.* Let $K$ be the function field of $X$, or of $\mathcal{A}$. The embeddings $\sigma : K \hookrightarrow F$ over $k$, in general position with respect to $L$ (i.e., with $\text{tr.deg}(\sigma(K)L/L) = \text{tr.deg}(K/k)$), form a single $U_L$-orbit. By Corollary 3.5, for any generic point $w : K \overset{/k}{\hookrightarrow} F$ in general position with respect to $L$ the $\mathbb{Q}$-space $W$ is generated by $\tau w - w$ for all $\tau \in U_L$, so $H_0(U_L, W) = 0$. $\square$

## 6 The category $\mathcal{I}_G$

**Lemma 6.1.** *Let $W$ be a smooth representation of $G$, and $L$ be an extension of $k$ in $F$. Then $W^{U_L} = \bigcup_{L_0 \subseteq L} W^{U_{L_0}}$, where $L_0$ runs over extensions of $k$ of finite type.*

*Proof.* For any $w \in W^{U_L}$ there is an extension $L_1$ of $k$ of finite type such that $w \in W^{U_{L_1}}$, so $w \in W^H$, where $H = \langle U_L, U_{L_1} \rangle$.

Consider first the case $\text{tr.deg}(L/k) < \infty$. Let $L_2$ be generated over $L_1$ by a transcendence basis of $L$ over $k$. Then $H \supseteq \langle U_L, U_{L_2} \rangle$ and $\overline{L_2 \bigcap L} = \overline{L}$. One has the following evident inclusions $U_{\overline{L}} \subseteq \langle U_L, U_{L_2} \rangle =: H' \subseteq U_{L_2 \bigcap L}$. Consider the quotients $H'/U_{\overline{L}} = \langle \text{Gal}(\overline{L}/L), U_{L_2}/U_{L_2\overline{L}} \rangle$ and $U_{L_2 \bigcap L}/U_{\overline{L}} = \text{Gal}(\overline{L}/L_2 \bigcap L)$.

By the standard Galois theory (e.g., S.Lang, Algebra, Chapter VIII, §1, Theorem 4), $U_{L_2}/U_{L_2\overline{L}} = \text{Gal}(L_2\overline{L}/L_2) \cong \text{Gal}(\overline{L}/L_2 \bigcap \overline{L})$.

According to Lemma 2.1, $\langle \text{Gal}(\overline{L}/L_2 \bigcap \overline{L}), \text{Gal}(\overline{L}/L) \rangle = \text{Gal}(\overline{L}/L_2 \bigcap L)$, so $H'/U_{\overline{L}} = U_{L_2 \bigcap L}/U_{\overline{L}}$, and therefore, $H' = U_{L_2 \bigcap L}$. Finally, $H \supseteq U_{L_0}$, where $L_0 = L_2 \bigcap L$.

Now consider the case $\text{tr.deg}(L/k) = \infty$. The group $H$ contains the subgroup $\langle U_{\overline{L}}, U_{\overline{L_1}} \rangle$, which coincides, by Proposition 2.14, with $U_{\overline{L} \bigcap \overline{L_1}}$. Let $L_2$ be generated over $L_1$ by a transcendence basis $S$ of $\overline{L_1} \bigcap \overline{L}$ over $k$ (in

particular, $L_2 \subset \overline{L_1}$). Then one has embeddings $L_1 \subseteq L_2 \subseteq \overline{L_1}$, and therefore, $\overline{L_1} = \overline{L_2}$, and thus, $\overline{L_1} \bigcap \overline{L} = \overline{L_2} \bigcap \overline{L}$. Similarly, $k(S) \subseteq L_2 \bigcap \overline{L} \bigcap \overline{L_1} \subseteq \overline{L} \bigcap \overline{L_1}$, and therefore, $\overline{L_2 \bigcap (\overline{L} \bigcap \overline{L_1})} = \overline{L} \bigcap \overline{L_1} = \overline{L_2} \bigcap (\overline{L} \bigcap \overline{L_1})$. By Proposition 2.14, this implies that $\langle U_{L_2}, U_{\overline{L} \bigcap \overline{L_1}} \rangle = U_{L_2 \bigcap \overline{L} \bigcap \overline{L_1}}$, so $H$ contains $U_{L_3}$, where $L_3 = L_2 \bigcap \overline{L}$. Let $L_4$ be the minimal $\mathrm{Gal}(\overline{L}/L)$-invariant extension of $L_3$. As $L_3$ is a subfield of $\overline{L}$ finitely generated over $k$, $L_4$ is an extension of finite type of $L_3$. Then $\overline{L_4 \bigcap L} = \overline{L_4} = \overline{L_4} \bigcap \overline{L}$, so by Proposition 2.14, $\langle U_L, U_{L_4} \rangle = U_{L_0}$, where $L_0 = L_4 \bigcap L = L_4^{\mathrm{Gal}(\overline{L}/L)}$.

Finally, $H \supseteq U_{L_0}$, where $L_0$ is a subfield of $L$ of finite type over $k$. $\square$

**Corollary 6.2.** *Let $W$ be a smooth representation of $G$ such that $W^{U_{L_1}} = W^{U_{L_1(t)}}$ for any extension $L_1$ of $k$ of finite type and any $t \in F - \overline{L_1}$. Then $W^{U_L} = W^{U_{L'}}$ for any extension $L$ of $k$ and any purely transcendental extension $L'$ of $L$.*

*Proof.* By Lemma 6.1, $W^{U_{L'}} = \bigcup_{L_0 \subseteq L'} W^{U_{L_0}}$, where $L_0$ runs over extensions of $k$ of finite type. Let $L' = L(x_1, x_2, x_3, \dots)$ for some $x_1, x_2, x_3, \dots$ algebraically independent over $L$. Each $L_0$ of this type is a subfield in $L_1(x_1, \dots, x_N)$ for some $L_1 \subseteq L$ of finite type over $k$ and some integer $N \geq 0$. Then $W^{U_{L'}} = \bigcup_{L_0 \subseteq L'} W^{U_{L_0}} \subseteq \bigcup_{L_1 \subseteq L} W^{U_{L_1}} = W^{U_L}$, so $W^{U_{L'}} = W^{U_L}$. $\square$

**Lemma 6.3.** *Let $F' \subsetneqq F$ be an algebraically closed extension of $k$. Suppose that $W^{U_{F'}} = W^{U_{F'(x)}}$ for a smooth representation $W$ of $G$ and some $x \in F - F'$. Then $W^{U_L} = W^{U_{L(x)}}$ for any extension $L$ of $k$ inside $F'$.*

*Proof.* Fix a transcendence basis $S$ of $F'$ over $L$ and set $L_1 = L(S)$.

As the group $U_{L_1}$ is an extension of $G_{F'/L_1}$ by $G_{F/F'}$, and $U_{L_1(x)}$ is an extension of the group $G_{F'(x)/L_1(x)} = G_{F'/L_1}$ by $G_{F/F'(x)}$, one has

$$W^{U_{L_1}} = \left(W^{G_{F/F'}}\right)^{G_{F'/L_1}} = \left(W^{G_{F/F'(x)}}\right)^{G_{F'(x)/L_1(x)}} = W^{U_{L_1(x)}}.$$

For any $y \in S \cup \{x\}$ there is $\sigma \in U_L$ inducing a permutation of $S \cup \{x\}$ that transforms $(S \cup \{x\}) - \{y\}$ to $S$. Such $\sigma$ induces an automorphism of $W^{U_{L_1(x)}}$ transforming $W^{U_{L((S\cup\{x\})-\{y\})}}$ to $W^{U_{L_1}}$, and therefore, the latter two spaces coincide.

This implies that $W^{U_{L_1}}$ is fixed by the subgroup of $G$ generated by $U_{L((S\cup\{x\})-\{y\})}$ for all $y \in S \cup \{x\}$. By Lemma 2.16, this subgroup is dense in $U_L$, and therefore, $W^{U_{L_1}} = W^{U_L}$. $\square$

Let $\mathcal{I}_G$ be the full subcategory in $\mathcal{S}m_G$ consisting of those representations $W$ of $G$ for which $W^{G_{F/L}} = W^{G_{F/L'}}$ for any extension $L$ of $k$ in $F$ and any purely transcendental extension $L'$ of $L$ in $F$. For each integer $q \geq 0$ let $\mathcal{I}_G^q$ be the full subcategory in $\mathcal{I}_G$ consisting of those representations $W$ of $G$ for which $W^{G_{F/F'}} = 0$ for any algebraically closed $F'$ with $\mathrm{tr.deg}(F'/k) = q - 1$.

**Proposition 6.4.** *Any admissible representation of $G$ is an object in $\mathcal{I}_G$ if $n=\infty$.*

*Proof.* Let $W$ be an admissible representation of $G$, $L$ an extension of $k$ in $F$ of finite type and $x,y\in F$ are algebraically independent over $L$. Then the finite-dimensional space $W^{U_L}$ is included into the finite-dimensional spaces $W^{U_{L(x)}}$ and $W^{U_{L(y)}}$; and the latter ones are included into the finite-dimensional space $W^{U_{L(x,y)}}$. As the group $U_{L(x+y,xy)}$ is an extension of the group $\{1,\alpha\}=\mathrm{Gal}(L(x,y)/L(x+y,xy))$ (so $\alpha x=y$ and $\alpha y=x$) by $U_{L(x,y)}$, one has $W^{U_{L(x+y,xy)}}=\left(W^{U_{L(x,y)}}\right)^{\langle\alpha\rangle}$. As the subgroups $U_{L(x+y,xy)}$ and $U_{L(x,y)}$ are conjugated in $G$, the spaces $W^{U_{L(x+y,xy)}}$ and $W^{U_{L(x,y)}}$ are of the same dimension. This implies that $W^{U_{L(x+y,xy)}}=W^{U_{L(x,y)}}$, and thus, $\alpha$ acts trivially on $W^{U_{L(x,y)}}$.

Notice, however, that $\alpha$ permutes $W^{U_{L(x)}}$ and $W^{U_{L(y)}}$, so $W^{U_{L(x)}}=W^{U_{L(y)}}$. By Lemma 2.16, the group generated by $U_{L(x)}$ and $U_{L(y)}$ is dense in $U_L$, and therefore, $W^{U_L}=W^{U_{L(x)}}$. $\square$

**Corollary 6.5.** *The category of admissible representations of $G$ over $E$ is abelian. It is closed under extensions in $\mathcal{S}m_G(E)$.*

*Proof.* It suffices to check that for any short exact sequence

$$0\longrightarrow W_1\longrightarrow W_2\longrightarrow W_3\longrightarrow 0$$

of representations of $G$ with admissible $W_2$ the representations $W_1$ and $W_3$ of $G$ are also admissible. For any subextension $L$ of finite type over $k$ and a transcendence basis $t_1,t_2,t_3,\dots$ of $F$ over $L$ set $L'=L(t_1,t_2,t_3,\dots)$. Then the sequence $0\longrightarrow W_1^{U_{L'}}\longrightarrow W_2^{U_{L'}}\longrightarrow W_3^{U_{L'}}\longrightarrow 0$ is exact. If $n<\infty$, we can assume that $F=\overline{L}$. As $W_2\in\mathcal{I}_G(E)$ in the case $n=\infty$, the middle term coincides with $W_2^{U_L}$, which is a finite-dimensional space, and therefore, so are the terms $W_1^{U_{L'}}$ and $W_3^{U_{L'}}$, containing $W_1^{U_L}$ and $W_3^{U_L}$, respectively. This implies that $W_1$ and $W_3$ are admissible. $\square$

Remark. If $n=\infty$, the category of smooth representations of $G$ has such disadvantage that it has no non-zero projective objects.

*Proof.* Let $W$ be a projective object in the category of smooth $E$-representations of $G$. Choose a system of generators $\{e_j\}_{j\in J}$ of $W$. This determines a surjection

$$\bigoplus_{j\in J}E[G/U_{L_j}]\xrightarrow{\pi}W,$$

where $U_{L_j}\subseteq\mathrm{Stab}_{e_j}$. Fix an element $i_0\in J$ and for each $j\in J$ fix an extension $L'_j$ of $L_j$ such that $\mathrm{tr.deg}(L'_j/k)>\mathrm{tr.deg}(L_{i_0}/k)$. As $W$ is projective, the composition of $\pi$ with the surjection $\bigoplus_{j\in J}E[G/U_{L'_j}]\longrightarrow\bigoplus_{j\in J}E[G/U_{L_j}]$ splits, and therefore, there is an element in $\bigoplus_{j\in J}E[G/U_{L'_j}]$ with the same stabilizer as $e_{i_0}$. However, as $E[G/U_{L'_j}]^{U_{L_{i_0}}}=0$, this implies that $e_{i_0}=0$, and thus, $W=0$. $\square$

**Lemma 6.6.** *The functor $H^0(G_{F/L},-):\mathcal{I}_G\longrightarrow\mathcal{V}ect_{\mathbb{Q}}$ is exact for any extension $L$ of $k$ in $F$. $\mathcal{I}_G$ is closed under taking subquotients in $\mathrm{Sm}_G$.*

*$\{\mathcal{I}^q_G\}_{q\geq 0}$ is a decreasing filtration of the category $\mathcal{I}_G$ by Serre subcategories.*[11]

*Proof.* By definition, the functors $H^0(G_{F/L},-)$ and $H^0(G_{F/L'},-)$ coincide on $\mathcal{I}_G$ for any purely transcendental extension $L'$ of $L$. The group $G_{F/L'}$ is compact, if $F$ is algebraic over $L'$, so $H^0(G_{F/L'},-)$ is exact on $\mathcal{S}m_G$, and thus, its restriction to $\mathcal{I}_G$ is also exact.

Let $W\in\mathcal{I}^q_G$ and $W_1\subseteq W$ a subrepresentation of $G$. Then for any extension $L$ of $k$ in $F$ and any purely transcendental extension $L'$ of $L$ in $F$ one has $W_1^{G_{F/L}}=W_1\bigcap W^{G_{F/L}}=W_1\bigcap W^{G_{F/L'}}=W_1^{G_{F/L'}}$, so $W_1\in\mathcal{I}_G$. Now it is clear that $W_1\in\mathcal{I}^q_G$.

As $V^{G_{F/L}}\subseteq V^{G_{F/L'}}$ for any representation $V$ of $G$, to show that $V:=W/W_1\in\mathcal{I}_G$ we may suppose that $F$ is algebraic over $L'$. Then $V^{G_{F/L}}\subseteq V^{G_{F/L'}}=W^{G_{F/L'}}/W_1^{G_{F/L'}}=W^{G_{F/L}}/W_1^{G_{F/L}}\subseteq V^{G_{F/L}}$, so $V^{G_{F/L}}=V^{G_{F/L'}}$. It follows from the exactness of $H^0(G_{F/F'},-)$ that $V\in\mathcal{I}^q_G$.

Let $0\longrightarrow W_1\longrightarrow E\xrightarrow{\beta}W_2\longrightarrow 0$ be a short exact sequence in $\mathcal{I}_G$ with $W_1,W_2\in\mathcal{I}^q_G$. Then for any algebraically closed $F'$ with $\mathrm{tr.deg}(F'/k)=q-1$ the restriction of $\beta$ to $E^{G_{F/F'}}$ factors through $W_2^{G_{F/F'}}=0$, so $E^{G_{F/F'}}\subseteq W_1\bigcap E^{G_{F/F'}}=0$, and therefore, $E\in\mathcal{I}^q_G$. □

**Lemma 6.7.** *If $F'$ is a subfield in $F$ with $\mathrm{tr.deg}(F'/k)=\infty$ then the functor $H^0(G_{F/\overline{F'}},-)$ from $\mathcal{S}m_G$ to $\mathcal{S}m_{G_{\overline{F'}/k}}$ is an equivalence of categories (inducing an equivalence of $\mathcal{I}_G$ and $\mathcal{I}_{G_{\overline{F'}/k}}$). The functor $H^0(G_{F/K},-)$ from $\mathcal{S}m_G$ to $\mathcal{V}ect_{\mathbb{Q}}$ is exact if and only if $\mathrm{tr.deg}(K/k)=\mathrm{tr.deg}(F/k)(\leq\infty)$.*

*Proof.* There exists a field isomorphism $\varphi:F\xrightarrow{\sim}\overline{F'}$ identical on $k$. Then $\varphi$ induces an isomorphism of topological groups $G_{\overline{F'}/k}\xrightarrow{\sim}G$ by $\tau\longmapsto\varphi^{-1}\tau\varphi$ and an equivalence of the categories of representations of $G$ and representations of $G_{\overline{F'}/k}$ by $\pi\longmapsto\varphi^*\pi$, where $\varphi^*\pi(\tau)=\pi(\varphi^{-1}\tau\varphi)$.

For any subfield $L\subset F$ finitely generated over $k$ there exists an element $\sigma\in G$ such that $\sigma|_L=\varphi|_L$. Let $W$ be a smooth representation of $G$. Then $\varphi$ and $\sigma$ induce the same isomorphism $W^{U_L}\xrightarrow{\sim}W^{U_{\sigma(L)}}=W^{U_{\varphi(L)}}$. Passing to the direct limit with respect to $L$, we get an isomorphism $W\xrightarrow{\sim}W^{G_{F/\overline{F'}}}$. For any $\tau\in G_{\overline{F'}/k}$ and any $w\in W$ one has $\varphi\pi(\varphi^{-1}\tau\varphi)w=\tau\varphi w$ (since $\varphi$ is a limit of elements of $G$), i.e., $\varphi^*\pi\cong W^{G_{F/\overline{F'}}}$.

Now, if $\mathrm{tr.deg}(K/k)=\infty$, then $H^0(G_{F/K},-)$ is the composition of exact functors $H^0(G_{F/\overline{K}},-)$ and $H^0(\mathrm{Gal}(\overline{K}/K),-)$. Otherwise, if $\mathrm{tr.deg}(K/k)<\mathrm{tr.deg}(L/k)$, then the only $G_{F/\overline{K}}$-invariant element in $\mathbb{Q}[G/U_L]$ is zero, so $H^0(G_{F/K},-)$ transforms the surjection $\mathbb{Q}[G/U_L]\longrightarrow\mathbb{Q}$ to $0\longrightarrow\mathbb{Q}$. □

[11] A full subcategory of an abelian category $\mathcal{A}$ is called a Serre subcategory if it is stable under taking subquotients and extensions in $\mathcal{A}$.

*6.1 The functor $\mathcal{I}$*

For a representation $M$ of $G$ define $N_jM$ as the subspace generated by the invariants $M^{G_{F/F_j}}$ for all subfields $F_j \subseteq F$ with $\mathrm{tr.deg}(F_j/k) = j$. Clearly, $N_jM \subseteq N_{j+1}M$, $M = \bigcup_{j\geq 0} N_jM$ if $M$ is smooth,

- $N_jM$ is the subrepresentation $M$ of $G$ in $M$ generated by $M^{G_{F/F_j}}$ for some algebraically closed $F_j$,
- restriction to $N_jM$ of each $G$-homomorphism $M \longrightarrow M'$ factors through $N_jM'$;
- $N_{i+j}(M_1 \otimes M_2) \supseteq N_iM_1 \otimes N_jM_2$.

**Proposition 6.8.** *For any object $W \in \mathcal{S}m_G$ and any integer $q \geq 0$ there is its quotient $\mathcal{I}^qW \in \mathcal{I}^q_G$ such that any $G$-homomorphism from $W$ to an object of $\mathcal{I}^q_G$ factors through $\mathcal{I}^qW$. The functor*[12] *$\mathcal{S}m_G \xrightarrow{\mathcal{I}^q} \mathcal{I}^q_G$ given by $W \longmapsto \mathcal{I}^qW$ is right exact and $\mathcal{I}^qW = \mathcal{I}W/N_{q-1}\mathcal{I}W$.*

*Proof.* Let $W' \in \mathcal{I}^q_G$. Any $G$-homomorphism $W \xrightarrow{\alpha} W'$ factors through $\alpha(W)$, which is an object in $\mathcal{I}^q_G$, so we may assume that $\alpha$ is surjective. Let $L$ be an extension of $k$ in $F$ and $L'$ a purely transcendental extension of $L$ in $F$ over which $F$ is algebraic. As the functor $H^0(U_{L'}, -)$ is exact on $\mathcal{S}m_G$, the morphism $\alpha$ induces a surjection $W^{U_{L'}} \longrightarrow (W')^{U_{L'}}$. As $(W')^{U_L} = (W')^{U_{L'}}$, the subgroup $U_L$ acts trivially on $(W')^{U_{L'}}$, and therefore, the subrepresentation $W_L = \langle \sigma w - w \mid \sigma \in U_L,\ w \in W^{U_{L'}}\rangle_G$ of $G$ is in the kernel of $\alpha$ (it is independent of $L'$ as all possible $L'$ form a single $U_L$-orbit and $\sigma\tau w - \tau w = (\sigma\tau)w - w - (\tau w - w)$ for any $\tau \in U_L$ and any $w \in W^{U_{L'}}$; moreover, $W_L$ depends only on the $G$-orbit of $L$). This implies that $\alpha$ factors through $\mathcal{I}W := W/\sum_L W_L$.

The representation $\mathcal{I}W$ of $G$ is smooth, so the map $W^{U_{L'}} \longrightarrow (\mathcal{I}W)^{U_{L'}}$ induced by the projection is surjective, and therefore, one can lift any element $\overline{w} \in (\mathcal{I}W)^{U_{L'}}$ to an element $w \in W^{U_{L'}}$. Then $\sigma\overline{w} - \overline{w}$ coincides with the projection of $\sigma w - w$ for any $\sigma \in U_L$. Note, that $\sigma w - w \in W_L$, so its projection is zero, and therefore, $\sigma\overline{w} = \overline{w}$ for any $\sigma \in U_L$. As $(\mathcal{I}W)^{U_L} \subseteq (\mathcal{I}W)^{U_{L'}}$, this means that $(\mathcal{I}W)^{U_L} = (\mathcal{I}W)^{U_{L'}}$, and thus, $\mathcal{I}W \in \mathcal{I}_G$.

We may further suppose that $W \in \mathcal{I}_G$. As $N_{q-1}W' = 0$ and $N_{q-1}$ is functorial, $N_{q-1}W$ is in the kernel of $W \xrightarrow{\alpha} W'$, so $\alpha$ factors through $\mathcal{I}^qW := W/N_{q-1}W$.

As the functor $H^0(G_{F/F'}, -)$ is exact on $\mathcal{I}_G$, one has a short exact sequence

$$0 \longrightarrow (N_{q-1}W)^{G_{F/F'}} \longrightarrow W^{G_{F/F'}} \longrightarrow (\mathcal{I}^qW)^{G_{F/F'}} \longrightarrow 0,$$

where $(N_{q-1}W)^{G_{F/F'}} = W^{G_{F/F'}}$ for any algebraically closed $F'$ of transcendence degree $q-1$ over $k$, so $(\mathcal{I}^qW)^{G_{F/F'}} = 0$, i.e., $\mathcal{I}^qW \in \mathcal{I}^q_G$.

[12] cf. [GM], Chapter II, §3.23

As $\mathrm{Hom}_{\mathcal{I}^q_G}(\mathcal{I}^qW, W') = \mathrm{Hom}_{\mathcal{S}m_G}(W, W')$ for any $W \in \mathcal{S}m_G$ and $W' \in \mathcal{I}^q_G$, i.e., $\mathcal{I}^q$ is left adjoint to the identity functor $\mathcal{I}^q_G \hookrightarrow \mathcal{S}m_G$, it is right exact (cf., e.g., [GM], Chapter II, §6.20). $\square$

Example. Let $\mathcal{A}$ be a one-dimensional group scheme over $k$, $m \geq 1$ an integer, and $W = N_qW$ be a smooth representation of $G$, where $q \leq n-1$. Then $\mathcal{I}(S^m\mathcal{A}(F) \otimes W) = 0$ if either $m$ is even, or if $\mathcal{A} = \mathbb{G}_a$, or if $\mathcal{A} = \mathbb{G}_m$. In particular, the natural projection $\mathcal{A}(F)^{\otimes N}_{\mathbb{Q}} \longrightarrow \bigwedge^N \mathcal{A}(F)_{\mathbb{Q}}$ induces an isomorphism $\mathcal{I}(\mathcal{A}(F)^{\otimes N}_{\mathbb{Q}}) \longrightarrow \mathcal{I}(\bigwedge^N \mathcal{A}(F)_{\mathbb{Q}})$ if $n \geq N-1$.

*Proof.* For any vector space $V$ and its proper subspace $V'$ the space $S^mV$ is spanned by the elements $x^{\otimes m}$ for all $x \in V - V'$, so $S^m\mathcal{A}(F) \otimes W$ is spanned by the elements $x^{\otimes m} \otimes w$ for all $w \in W$ with $\mathrm{Stab}_w \supseteq U_{k'}$, $\mathrm{tr.deg}(k'/k) < q+1$, and all $x \in \mathcal{A}(F) - \mathcal{A}(k')$. If $m$ is even then the stabilizer of such $x^{\otimes m} \otimes w$ contains the group $\{\sigma \in U_{k'} \mid \sigma x = \pm x\}$, which contains the subgroup $U_{k'(t)}$ for an element $t \in k(x) := x(k(\mathcal{A}))$ with quadratic extension $k(x)/k(t)$. If $\mathcal{A}$ is rational then the stabilizer of $x^{\otimes m} \otimes w$ contains the subgroup $U_{k'(x)}$. This implies that the image of $x^{\otimes m} \otimes w$ in $\mathcal{I}(S^m\mathcal{A}(F) \otimes W)$ spans a trivial subrepresentation of $U_{k'}$. On the other hand, as $n \geq q+1 > \mathrm{tr.deg}(k'/k)$, there is $\sigma \in U_{k'}$ such that $\sigma x = 2 \cdot x$, and thus, $\sigma(x^{\otimes m} \otimes w) = 2^m \cdot (x^{\otimes m} \otimes w)$, which means that this trivial subrepresentation of $U_{k'}$ is zero, so $\mathcal{I}(S^m\mathcal{A}(F) \otimes W) = 0$. $\square$

Remark. $\mathcal{I}$ is not left exact, e.g., it transforms the injection $k \hookrightarrow F$ to $k \longrightarrow 0$.

*Conjecture 6.9.* If $n = \infty$ then for any $j \geq 0$ and any object $W$ of $\mathcal{I}_G$ the $G$-module $gr^N_jW$ is semi-simple.

This is evident when $j = 0$, and deduced easily from Corollary 6.22 when $j = 1$.

**Corollary 6.10.** *If $n = \infty$ and Conjecture 6.9 holds for any $0 \leq j \leq q-1$ then the functor $\mathcal{I}^q$ is exact on $\mathcal{I}_G$. This is equivalent to the strict compatibility of the filtration $N_0 \subseteq \cdots \subseteq N_{q-1}$ with morphisms in $\mathcal{I}_G$.*

*Proof.* Let $0 \longrightarrow W_1 \xrightarrow{\alpha} W_2 \xrightarrow{\beta} W_3 \longrightarrow 0$ be a short exact sequence in $\mathcal{I}_G$. Then the first term of the induced exact sequence

$$0 \longrightarrow (W_1 \bigcap N_{q-1}W_2)/N_{q-1}W_1 \longrightarrow \mathcal{I}^qW_1 \longrightarrow \mathcal{I}^qW_2 \longrightarrow \mathcal{I}^qW_3 \longrightarrow 0$$

measures deviation from the strict compatibility of $N_{q-1}$ with the morphism $\alpha$.

To show that the filtration $N_\bullet$ on objects of $\mathcal{I}_G$ is strictly compatible with the morphisms, i.e., $\varphi(N_jW_1) = \varphi(W_1) \bigcap N_jW_2$ for any morphism $W_1 \xrightarrow{\varphi} W_2$ in $\mathcal{I}_G$, we proceed by induction on $j \geq 0$. This is clear when $j = 0$. By Lemma 6.6, $\varphi(W_1^{G_{F/F'}}) = \varphi(W_1)^{G_{F/F'}}$ for any subfield $F'$ with $\mathrm{tr.deg}(F'/k) = j$, so $\varphi(N_jW_1) = N_j\varphi(W_1)$, and thus,

we may further suppose that $\varphi$ is an embedding: $W_1 \subset W_2$. Suppose that $W_1 \bigcap N_jW_2 \neq N_jW_1$. Replace $W_1$ by $W_1' := \frac{W_1 \bigcap N_jW_2}{N_jW_1}$ and $W_2$ by $W_2' := \frac{N_jW_2}{N_jW_1}$. Then $0 \neq W_1' \subset W_2' = N_jW_2'$ and $N_jW_1' = 0$. (By Proposition 6.8, $W_1' \subseteq \mathcal{I}^{j+1}W_1$, so by Lemma 6.6, $W_1' \in \mathcal{I}_G^{j+1}$, which is equivalent to $N_jW_1' = 0$.)

The induction assumption excludes the case $W_1' \subseteq N_{j-1}W_2'$, so we may replace $W_1'$ by $W_1 := \frac{W_1'}{N_{j-1}W_2'}$ and $W_2'$ by $W_2 := gr_j^N W_2'$. Then $0 \neq W_1 \subset W_2 = N_jW_2$ and $N_jW_1 = N_{j-1}W_2 = 0$.

Assuming Conjecture 6.9, there is a morphism $W_2 \xrightarrow{\xi} W_1$ splitting the embedding $W_1 \subset W_2$. However, $\xi(W_2) = \xi(N_jW_2) \subseteq N_jW_1 = 0$, leading to contradiction. $\square$

Remark. The inclusion $\mathbb{Q}[G/U_L]^\circ \hookrightarrow \mathbb{Q}[G/U_L]$ is an example of morphism of smooth $G$-modules which is *not* strictly compatible with the filtration $N_\bullet$: $N_{\mathrm{tr.deg}(L/k)}\mathbb{Q}[G/U_L]^\circ$ coincides with

$$\left\{ \sum_{[\sigma]\in G/U_L} a_\sigma[\sigma] \;\middle|\; \begin{array}{l} \text{for any } F' \text{ with } \mathrm{tr.deg}(F'/k) = \mathrm{tr.deg}(L/k) \\ \text{one has } \sum_{\sigma(L)\subset\overline{F'}} a_\sigma = 0 \end{array} \right\}$$

which is different from $\mathbb{Q}[G/U_L]^\circ$, but $\mathbb{Q}[G/U_L] = N_{\mathrm{tr.deg}(L/k)}\mathbb{Q}[G/U_L]$.

**Corollary 6.11.** *Let $s \geq 0$ be an integer and $L$ be an extension of $k$ in $F$ of finite type with* $\mathrm{tr.deg}(L/k) = q$. *Then $C_L/N_{s-1} := \mathcal{I}^s\mathbb{Q}[G/U_L]$ is a projective object in $\mathcal{I}_G^s$ left orthogonal to $\mathcal{I}_G^{q+1}$. In particular, there are sufficiently many projective objects in $\mathcal{I}_G^s$ for any $s \geq 0$.*

*Proof.* Any exact sequence $0 \longrightarrow W_1 \longrightarrow W_2 \longrightarrow W_3 \longrightarrow 0$ in $\mathcal{I}_G^s$ gives an exact sequence $0 \longrightarrow W_1^{U_L} \longrightarrow W_2^{U_L} \longrightarrow W_3^{U_L} \longrightarrow 0$, where $W_j^{U_L} = \mathrm{Hom}_G(\mathbb{Q}[G/U_L], W_j) = \mathrm{Hom}_G(C_L/N_{s-1}, W_j)$. $\square$

**Lemma 6.12.** *For any $1 \leq n \leq \infty$, any subfield $L_1 \subset F$ of finite type over $k$, and any a unirational extension $L_2$ of $L_1$ in $F$ of finite type there is a natural isomorphism $C_{L_2} \xrightarrow{\sim} C_{L_1}$.*

*Proof.* Let $x \in F - \overline{L_1}$ and let $\mathbb{Q}[G/U_{L_1(x)}] \xrightarrow{\alpha} \mathcal{I}\mathbb{Q}[G/U_{L_1(x)}] = C_{L_1(x)}$ be the projection. For any $\sigma \in U_{L_1}$ one has $[\sigma U_{L_1(x)}] - [U_{L_1(x)}] = \sigma[U_{L_1(x)}] - [U_{L_1(x)}] \in W_{L_1} \subset \mathbb{Q}[G/U_{L_1(x)}]$ (see definition of $\mathcal{I}$ in the proof of Proposition 6.8), and thus, $\alpha$ factors through $\mathbb{Q}[G/U_{L_1}]$, and therefore, through $\mathcal{I}\mathbb{Q}[G/U_{L_1}] = C_{L_1}$, i.e., the surjection $\mathbb{Q}[G/U_{L_1(x)}] \longrightarrow \mathbb{Q}[G/U_{L_1}]$ induces an isomorphism $C_{L_1(x)} \longrightarrow C_{L_1}$.

One has $L_1 \subseteq L_2 \subseteq L_1(x_1, \dots, x_N)$ for some $x_1, \dots, x_N$ algebraically independent over $L_1$. Then the surjections

$$\mathbb{Q}[G/U_{L_1(x_1,\dots,x_N)}] \longrightarrow \mathbb{Q}[G/U_{L_2}] \longrightarrow \mathbb{Q}[G/U_{L_1}]$$

induce surjections $C_{L_1(x_1,\dots,x_N)} \longrightarrow C_{L_2} \longrightarrow C_{L_1}$, where the composition is an isomorphism. $\square$

**Lemma 6.13.** *Let $k \subseteq L \subset F'$ be subfields in $F$. Then*

$$\mathbb{Q}[G/G_{F/F'}] \otimes_{\mathbb{Q}[G_{F'/k}]} \mathbb{Q}[G_{F'/k}/G_{F'/L}] = \mathbb{Q}[G/G_{\{F,F'\}/L}].$$

*Proof.* The module on the left coincides with $\mathbb{Q}[G/G_{F/F'}]/\langle[\sigma\tau]-[\sigma] \mid \sigma \in G, \tau \in G_{\{F,F'\}/L}\rangle$, which is the same as $\mathbb{Q}[G/G_{\{F,F'\}/L}]$. $\square$

**Lemma 6.14.** *If $L$ is a finitely generated extension of $k$ and $F'$ is an algebraically closed subfield in $F$ with* $\mathrm{tr.deg}(F'L/L) = \mathrm{tr.deg}(F'/F' \bigcap L) < \infty$ *then there is a canonical isomorphism $\mathcal{I}E[G/G_{\{F,F'\}/L}] \xrightarrow{\sim} C_L \otimes E$.*

*Proof.* Let $t_1, \dots, t_N$ be a transcendence basis of $F'$ over $F' \bigcap L$, and thus, a transcendence basis of $F'L$ over $L$. Then the surjections

$$\mathbb{Q}[G/U_{L(t_1,\dots,t_N)}] \longrightarrow \mathbb{Q}[G/G_{\{F,F'\}/L}] \longrightarrow \mathbb{Q}[G/U_L]$$

induce surjections $C_{L(t_1,\dots,t_N)} \longrightarrow \mathcal{I}\mathbb{Q}[G/G_{\{F,F'\}/L}] \longrightarrow C_L$. By Lemma 6.12, their composition is an isomorphism, so both arrows are isomorphisms. $\square$

**Proposition 6.15.** *$\mathcal{I}_G(E)$ is a Serre subcategory in $\mathcal{S}m_G(E)$ if $n = \infty$.*

*Proof.* By Lemma 6.6, the category $\mathcal{I}_G(E)$ is closed under taking subquotients in $\mathcal{S}m_G(E)$, so we have only to check that it is closed under extensions in $\mathcal{S}m_G(E)$.

Let $0 \longrightarrow W_1 \longrightarrow W \longrightarrow W_2 \longrightarrow 0$ be an extension in $\mathcal{S}m_G(E)$ with $W_1, W_2 \in \mathcal{I}_G(E)$. By Corollary 6.2 and Lemma 6.3, it suffices to check that $W^{U_{\overline{L}}} = W^{U_{L'}}$ for any extension $L$ of $k$ in $F$ of finite type and a purely transcendental extension $L'$ of $\overline{L}$ with $\overline{L'} = F$. As the functor $H^0(U_{L'}, -) = \mathrm{Hom}_{\mathcal{S}m_{U_{L'}}(E)}(E, -)$ is exact on $\mathcal{S}m_G(E)$ and $W_1, W_2 \in \mathcal{I}_G(E)$, this is equivalent to the vanishing of $\mathrm{Ext}^1_{\mathcal{S}m_{U_{\overline{L}}}(E)}(E, -)$ on $\mathcal{I}_G(E)$. As the forgetful functor $\mathcal{S}m_G(E) \longrightarrow \mathcal{S}m_{U_{\overline{L}}}(E)$ induces $\mathcal{I}_G(E) \longrightarrow \mathcal{I}_{U_{\overline{L}}}(E)$, one can replace $k$ with $\overline{L}$ and then it remains to show the vanishing of $\mathrm{Ext}^1_{\mathcal{S}m_G(E)}(E, -)$ on $\mathcal{I}_G(E)$.

Let $0 \longrightarrow W_1 \longrightarrow W \longrightarrow E \longrightarrow 0$ be an extension in $\mathcal{S}m_G(E)$ with $W_1 \in \mathcal{I}_G(E)$. Choose some $e \in W$ projecting to $1 \in E$. The stabilizer of $e$ contains an open subgroup $U_L$. Fix a maximal purely transcendental extension $L'$ of $k$ in $L$. Let $L''$ be a Galois extension of $L'$ containing $L$. Then $\frac{1}{[L'':L']} \sum_{\sigma \in \mathrm{Gal}(L''/L')} \sigma e$ is fixed by $U_{L'}$ and projects to $1 \in E$, so $W$ is a sum of $W_1$ and a quotient of $E[G/U_{L'}]$. Restriction to $E[G/U_{L'}]^\circ$ of the projection $E[G/U_{L'}] \longrightarrow W$ factors through $W_1$ which is an object in $\mathcal{I}_G(E)$.

Let $\sigma \in G$ be such an element that $L'$ and $\sigma(L')$ are in general position and $\sigma^2|_{L'} = id$. Then $[1] - [\sigma]$ is a generator of $E[G/U_{L'}]^\circ$, so there is a surjection $E[G/U_{L'\sigma(L')}] \longrightarrow E[G/U_{L'}]^\circ$ sending $[\tau]$ to $[\tau] - [\tau\sigma]$, and therefore, a surjection $E = C_{L'\sigma(L')} \otimes E \longrightarrow \mathcal{I}E[G/U_{L'}]^\circ$. However, $\sigma$ changes sign of the generator $[1]-[\sigma]$ of $E[G/U_{L'}]^\circ$, and thus, $\mathcal{I}E[G/U_{L'}]^\circ = 0$. This means that the projection $E[G/U_{L'}] \longrightarrow W$ factors through $E$, i.e., $W \cong W_1 \oplus E$. $\square$

*Conjecture 6.16.* If $n = \infty$ then $CH_0(X_F)_{\mathbb{Q}} = C_{k(X)}$ for any smooth irreducible proper variety $X$ over $k$.

Remark. This is true, by Corollary 6.21 and Lemma 6.12, if $k(X)$ is unirational over a one-dimensional field. Another example is given by $k(X) = k(x_1, \dots, x_m)^{\langle e_1 e_2^2 \cdots e_m^m \rangle}$, where $\sum_{j=1}^m x_j^d = 1$, $m$ is odd and $d \in \{m+1, m+2\}$, $e_i x_j = \zeta^{\delta_{ij}} \cdot x_j$ for a primitive $d^{\text{th}}$ root of unity $\zeta$. (The proof is the same as for $CH_0(X) = \mathbb{Z}$.)

**Proposition 6.17.** *If $n = \infty$ then for any irreducible variety $X$ over $k$ the kernel of $\mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}] \longrightarrow C_{k(X)}$ is the sum over all curves $y \in (k(X) \otimes_k F)_1$ of subspaces spanned by those linear combinations of generic (with respect to a field of definition of $y$) $F$-points of $\overline{\{y\}}$ that are linearly equivalent to zero on any compactification of $\overline{\{y\}}$.*

*Proof.* Let $\mathfrak{H}$ be the set of algebraically closed extensions $F_\infty$ of $k$ in $F$ such that $\text{tr.deg}(F/F_\infty) = 1$. Set $h_K := h_{U_K}$. By Lemma 6.3 and Corollary 6.2, the kernel ${}_\infty W$ of $W \longrightarrow \mathcal{I}W$ coincides with

$$\sum_{F_\infty \in \mathfrak{H}} \langle \sigma w - w \mid w \in W^{G_{F/F_\infty(t)}}, \sigma \in G_{F/F_\infty}, t \in F - F_\infty \rangle_{\mathbb{Q}}$$
$$= \sum_{F_\infty \in \mathfrak{H}} \langle \sigma h_{F_\infty(t)} w - h_{F_\infty(t)} w \mid w \in W, \sigma \in G_{F/F_\infty}, t \in F - F_\infty \rangle_{\mathbb{Q}}.$$

If $W = \mathbb{Q}[G/U_L]$ this is the same as

$$\sum_{F_\infty \in \mathfrak{H}} \langle \sigma h_{F_\infty(t)} \xi - h_{F_\infty(t)} \xi \mid \xi : L \overset{/k}{\hookrightarrow} F, \sigma \in G_{F/F_\infty}, t \in F - F_\infty \rangle_{\mathbb{Q}}.$$

We may suppose that in this sum $\xi(L) \not\subset F_\infty$, as otherwise $\sigma h_{F_\infty(t)} \xi - h_{F_\infty(t)} \xi = 0$. Then the pair $(\xi, F_\infty)$ determines the $F_\infty$-curve $C^{F_\infty, \xi} := \mathbf{Spec}((\xi \cdot id)(L \otimes_k F_\infty))$ on $\mathbf{Spec}(L \otimes_k F_\infty)$.

As for any pair of elements $\sigma \in G_{F/F_\infty}$ and $\alpha \in G_{F/F_\infty(t)}$ the homomorphism $L \otimes_k F_\infty \xrightarrow{\sigma\alpha\xi \cdot id} F$ is the composition of $L \otimes_k F_\infty \xrightarrow{\xi \cdot id} F$ with the automorphism $\sigma\alpha$, the kernels of $\sigma\alpha\xi \cdot id$ and of $\xi \cdot id$ coincide, and thus, $\sigma h_{F_\infty(t)} \xi - h_{F_\infty(t)} \xi$ is an $F$-divisor on $C^{F_\infty, \xi}$, which is a linear combination of generic $F$-points of $C^{F_\infty, \xi}$.

The triplet $(\xi, F_\infty, t)$ determines the $F_\infty$-curve $C_t^{F_\infty, \xi} := \mathbf{Spec}((\xi \cdot id)(L \otimes_k F_\infty[t]))$ endowed with the projections $\mathbb{A}^1_{F_\infty} \xleftarrow{T} C_t^{F_\infty, \xi} \xrightarrow{\beta} C^{F_\infty, \xi}$ induced by the inclusions $F_\infty[t] \subset (\xi \cdot id)(L \otimes_k F_\infty[t]) \supset (\xi \cdot id)(L \otimes_k F_\infty)$. Now one can rewrite $\sigma h_{F_\infty(t)} \xi - h_{F_\infty(t)} \xi$ as $\beta_* T^* \text{div}(\frac{T - \sigma t}{T - t})$, which is the divisor of a rational function on any compactification of $C^{F_\infty, \xi} \times_{F_\infty} F = \mathbf{Spec}((L \otimes_k F)/\mathfrak{p}) =: C_{\mathfrak{p}}$, where $\mathfrak{p} := \ker(L \otimes_k F \xrightarrow{\xi \otimes id} F \otimes_{F_\infty} F)$.

It remains to show that the divisor of a rational function $f$ on $C_{\mathfrak{p}}$, which is generic with respect to a field $F_0$ of definition of $C_{\mathfrak{p}}$, and independent of

any compactification, belongs to ${}_\infty W$. As the map $Z^1(F_0(\xi(L))\otimes_{F_0}F)\longrightarrow \mathcal{I}W$ factors through $\mathcal{I}_{F_0}Z^1(F_0(\xi(L))\otimes_{F_0}F)$, where $\xi$ is a generic $F$-point of the model of $C_{\mathfrak{p}}$ over $F_0$, it follows from Proposition 6.20 that $f$ is sent to zero, i.e., $\mathrm{div}(f)\in {}_\infty W$. $\square$

### *6.2 Objects of $\mathcal{I}_G$ of level 1*

For a subfield $L$ in $F$ of finite type over $k$ denote by $Z_0^{\mathrm{rat}}(L\otimes_k F)$ the kernel of the natural projection $\mathbb{Q}[G/U_L]\longrightarrow CH_0(X_F)_{\mathbb{Q}}$ for a smooth proper model $X$ of $L$ over $k$.

For any $W\in \mathcal{S}m_G$ one has a surjection $\bigoplus_{e\in W^{G_{F/F'}}}\langle e\rangle_G\longrightarrow N_1W$, where $F'$ is an algebraically closed extension of $k$ in $F$ with $\mathrm{tr.deg}(F'/k)=1$. This means that to describe the objects of $\mathcal{I}_G$ of level 1 it suffices to treat the case $W=\langle e\rangle_G$, where $\mathrm{Stab}_e\supseteq U_L$ with $L\cong k(X)$ for a smooth projective curve $X$ over $k$ of genus $g\geq 0$. Then $W$ is dominated by $C_L$. Let $J_X$ be the Jacobian of $X$.

**Lemma 6.18.** *If $n=\infty$ then the $G$-module $Z_0^{\mathrm{rat}}(L\otimes_k F)$ is generated by $w_N=\sum_{j=1}^N\sigma_j-\sum_{j=1}^N\tau_j$ for all $N>g$, where $(\sigma_1,\dots,\sigma_N;\tau_1,\dots,\tau_N)$ is a generic $F$-point of the fiber over $0$ of the map $X^N\times_k X^N\xrightarrow{p_N}\mathrm{Pic}^\circ X$ sending $(x_1,\dots,x_N;y_1,\dots,y_N)$ to the class of $\sum_{j=1}^N x_j-\sum_{j=1}^N y_j$.*

*Proof.* Let $\gamma_1,\dots,\gamma_s: L\overset{/k}{\hookrightarrow}F$ and $\delta_1,\dots,\delta_s: L\overset{/k}{\hookrightarrow}F$ be generic points of $X$ such that $\sum_{j=1}^s\gamma_j-\sum_{j=1}^s\delta_j$ is the divisor of a rational function on $X_F$.

We need to show that $\sum_{j=1}^s\gamma_j-\sum_{j=1}^s\delta_j$ belongs to the $G$-submodule in $Z_0^{\mathrm{rat}}(L\otimes_k F)$ generated by $w_N$'s.

There is a collection $\alpha_1,\dots,\alpha_g: L\overset{/k}{\hookrightarrow}F$ of generic points of $X$ such that the class of $\sum_{j=1}^s\gamma_j+\sum_{j=1}^g\alpha_j$ in $\mathrm{Pic}^{s+g}X$ is a generic point. Then there is a collection $\xi_1,\dots,\xi_{s+g}: L\overset{/k}{\hookrightarrow}F$ of generic points of $X$ in general position such that $\sum_{j=1}^s\gamma_j+\sum_{j=1}^g\alpha_j-\sum_{j=1}^{s+g}\xi_j$ is divisor of a rational function on $X_F$ (so the same holds also for $\sum_{j=1}^s\delta_j+\sum_{j=1}^g\alpha_j-\sum_{j=1}^{s+g}\xi_j$). We may, thus, suppose that $\delta_1,\dots,\delta_s$ are in general position.

Fix a collection $\varkappa_{ij}$ of generic points of $X$ in general position, also with respect to $\gamma_1,\dots,\gamma_s$ and to $\delta_1,\dots,\delta_s$, for $1\leq i\leq g$ and $1\leq j\leq s$ such that the classes of $\gamma_1+\sum_{i=1}^g\varkappa_{i1},\dots,\gamma_s+\sum_{i=1}^g\varkappa_{is}$ in $\mathrm{Pic}^{g+1}X$ are generic points in general position. Then one can choose a collection $\xi_{ij}$ of generic points of $X$ in general position for $0\leq i\leq g$ and $1\leq j\leq s$ such that $\gamma_j+\sum_{i=1}^g\varkappa_{ij}-\sum_{i=0}^g\xi_{ij}$ is divisor of a rational function on $X_F$ (so the same holds also for $\sum_{j=1}^s\sum_{i=0}^g\xi_{ij}-\left(\sum_{j=1}^s\delta_j+\sum_{j=1}^s\sum_{i=1}^g\varkappa_{ij}\right)$). We may, thus, suppose that both $\gamma_1,\dots,\gamma_s$ and $\delta_1,\dots,\delta_s$ are in general position.

Then there is a collection of generic points $\xi_1,\dots,\xi_s: L\overset{/k}{\hookrightarrow}F$ such that the points $(\gamma_1,\dots,\gamma_s;\xi_1,\dots,\xi_s)$ and $(\delta_1,\dots,\delta_s;\xi_1,\dots,\xi_s)$ are generic on $p_s^{-1}(0)$. Then $\sum_{j=1}^s\gamma_j-\sum_{j=1}^s\xi_j$ and $\sum_{j=1}^s\delta_j-\sum_{j=1}^s\xi_j$ are divisors of

rational functions on $X_F$. Clearly, such elements belong to the $G$-orbit of $w_s$. $\square$

**Lemma 6.19.** *The images of the generators $w_N = \sum_{j=1}^N \sigma_j - \sum_{j=1}^N \tau_j$ of $Z_0^{\mathrm{rat}}(L \otimes_k F)$ (from Lemma 6.18) in $W := \mathcal{I}Z_0^{\mathrm{rat}}(L \otimes_k F)$ are fixed by $G$.*

*Proof.* Set $M := N + g$. Then for any generic point $\gamma$ of $\mathrm{Pic}^M(X_F)$ in general position with respect to $\sigma_1, \dots, \sigma_N, \tau_1, \dots, \tau_N$ there is a collection $\alpha_1, \dots, \alpha_g$ of generic points of $X_F$, also in general position with respect to $\sigma_1, \dots, \sigma_N, \tau_1, \dots, \tau_N$, such that the class of $\sum_{j=1}^N \sigma_j + \sum_{j=1}^g \alpha_j$ coincides with $\gamma$.

There exists an $M$-tuple $\xi = (\xi_1, \dots, \xi_M)$ of generic points such that both $2M$-tuples $(\sigma_1, \dots, \sigma_N, \alpha_1, \dots, \alpha_g; \xi)$ and $(\tau_1, \dots, \tau_N, \alpha_1, \dots, \alpha_g; \xi)$ are generic points of the irreducible variety $p_M^{-1}(0)$.

The subfields

$$L_\xi := (\xi_1(L) \cdots \xi_M(L))^{\mathfrak{S}_M}, \quad L_\sigma := (\sigma_1(L) \cdots \sigma_N(L) \alpha_1(L) \cdots \alpha_g(L))^{\mathfrak{S}_M}$$

are isomorphic to the function field of the $M^{\mathrm{th}}$ symmetric power of $X$, and $L_\sigma L_\xi$ is isomorphic to the function field of the $\mathfrak{S}_M \backslash p_M^{-1}(0)/\mathfrak{S}_M$. As $M^{\mathrm{th}}$ symmetric power of $X$ is birational to the product of the Jacobian of $X$ and a projective space, the subfields $L_\xi$, $L_\sigma$, as well as $L_\sigma L_\xi$, are purely transcendental extensions of the subfield $\gamma(k(\mathrm{Pic}^M(X)))$ in $F$. Clearly, the same is true for the subfields $L_\tau := (\tau_1(L) \cdots \tau_N(L) \alpha_1(L) \cdots \alpha_g(L))^{\mathfrak{S}_M}$ and $L_\tau L_\xi$.

The elements $w_\sigma := \sum_{j=1}^N \sigma_j + \sum_{j=1}^g \alpha_j - \sum_{j=1}^M \xi_j$ and $w_\tau := \sum_{j=1}^N \tau_j + \sum_{j=1}^g \alpha_j - \sum_{j=1}^M \xi_j$ of $Z_0^{\mathrm{rat}}(L \otimes_k F)$ are fixed by $U_{L_\sigma L_\xi}$ and by $U_{L_\tau L_\xi}$, respectively. Then the classes of $w_\sigma$ and of $w_\tau$ in $W$ are fixed by $U_{\gamma(k(\mathrm{Pic}^M(X)))}$, and thus, so is their difference $w_N$.

Fix a purely transcendental extension $L_0$ of $k$ in $F$ with $\mathrm{tr.deg}(L_0/k) = g$ in general position with respect to $\sigma_1, \dots, \sigma_N, \tau_1, \dots, \tau_N$. Then

$$L_0 = \bigcap_{\gamma \in G,\ \gamma(k(\mathrm{Pic}^M(X))) \supset L_0} \gamma(k(\mathrm{Pic}^M(X))),$$

so by Lemma 2.1, the subgroup in $G$ generated by $U_{\gamma(k(\mathrm{Pic}^M(X)))}$ for such $\gamma$ contains $U_{L_0}$, and thus, the image of $w_N$ in $W$ is fixed by $U_{L_0}$.

Finally, as $W \in \mathcal{I}_G$, one has $w_N \in W^{U_{L_0}} = W^G$. $\square$

**Proposition 6.20.** *If $n = \infty$ then $\mathcal{I}Z_0^{\mathrm{rat}}(L \otimes_k F) = 0$.*

*Proof.* By Lemma 6.19, the images of the generators $w_N = \sum_{j=1}^N \sigma_j - \sum_{j=1}^N \tau_j$ of $Z_0^{\mathrm{rat}}(L \otimes_k F)$ (from Lemma 6.18) in $W := \mathcal{I}Z_0^{\mathrm{rat}}(L \otimes_k F)$ are fixed by $G$. As $(\sigma_1, \dots, \sigma_N; \tau_1, \dots, \tau_N)$ and $(\tau_1, \dots, \tau_N; \sigma_1, \dots, \sigma_N)$ are both generic points of the irreducible variety $p_N^{-1}(0)$ and the generic points form a single $G$-orbit, there is an element $\beta \in G$ such that $\beta\sigma_j = \tau_j$ and $\beta\tau_j = \sigma_j$ for all $1 \le j \le N$, so $\beta w = -w$. As $W$ is generated by the images of $w_N$'s, which are fixed by $G$, we get $W = 0$. $\square$

**Corollary 6.21.** *Let $X$ be a smooth projective curve over $k$, $L = k(X)$ be its function field and $\mathbb{Q}[G/U_L]^\circ$ be the group of degree-zero 0-cycles. If $n = \infty$ then $\mathcal{I}\mathbb{Q}[G/U_L]^\circ = \mathrm{Pic}^\circ(X_F)_\mathbb{Q}$ and $C_L = \mathrm{Pic}(X_F)_\mathbb{Q}$.*

*Proof.* Using Proposition 6.20, this can be done by applying the right-exact functor $\mathcal{I}$ to the short exact sequences $0 \longrightarrow Z_0^{\mathrm{rat}}(L \otimes_k F) \longrightarrow \mathbb{Q}[G/U_L]^\circ \longrightarrow CH_0(X_F)^0_\mathbb{Q} \longrightarrow 0$ and $0 \longrightarrow Z_0^{\mathrm{rat}}(L \otimes_k F) \longrightarrow \mathbb{Q}[G/U_L] \longrightarrow CH_0(X_F)_\mathbb{Q} \longrightarrow 0$. $\square$

**Corollary 6.22.** *If $n = \infty$ then any object of $\mathcal{I}_G$ of level 1 is a direct sum of a trivial module and a quotient of direct sum of modules $\mathcal{A}(F)_\mathbb{Q}$ for some abelian varieties $\mathcal{A}$ over $k$ by a trivial submodule.*

*Proof.* By Corollary 6.21, any object $W$ of $\mathcal{I}_G$ of level 1 is a quotient of $\bigoplus_{X \in I} \mathrm{Pic}(X_F)_\mathbb{Q}$ for a set $I$ of smooth projective curves over $k$. As there is a splitting $\mathrm{Pic}(X_F)_\mathbb{Q} \cong \mathbb{Q} \oplus \mathrm{Pic}^\circ(X_F)_\mathbb{Q}$, we get that $W$ is a quotient of $W^G \oplus \bigoplus_{\mathcal{A} \in J} \mathcal{A}(F)_\mathbb{Q}$ for a set $J$ of simple abelian varieties over $k$.

In particular, $W/W^G$ is semi-simple, so there is a subset $J' \subseteq J$ such that the projection $\bigoplus_{\mathcal{A} \in J'} W_\mathcal{A} \longrightarrow W/W^G$ is an isomorphism.

Then $W^G \oplus \bigoplus_{\mathcal{A} \in J'} \mathcal{A}(F)_\mathbb{Q} \longrightarrow W$ is a surjection with trivial kernel. $\square$

**Corollary 6.23.** *If $n = \infty$ then $\mathcal{A}(F)_\mathbb{Q}$ (resp., $W_\mathcal{A}$) is a projective object of $\mathcal{I}_G$ (resp., of $\mathcal{I}^1_G$) for any abelian $k$-variety $\mathcal{A}$.*

*Proof.* $\mathcal{A}(F)_\mathbb{Q}$ (resp., $W_\mathcal{A}$) is a direct summand of $\mathrm{Pic}(X_F)_\mathbb{Q}$ (resp., $W_{J_X}$) for a smooth curve $X$ on $\mathcal{A}$, which is a projective object in $\mathcal{I}_G$ (resp., in $\mathcal{I}^1_G$) by Corollary 6.21 and Corollary 6.11. $\square$

Define the following decreasing filtration on the objects of $\mathcal{I}_G$: $\mathcal{F}^j W = \bigcap_\varphi \ker \varphi$, where $\varphi$ runs over the set of $G$-homomorphisms from $W$ to objects of $\mathcal{I}_G$ of level $j$. If $N_\bullet$ is strictly compatible with the morphisms of $\mathcal{I}_G$ then one can assume that $\varphi$'s are surjective.

As $\ker(W \xrightarrow{id} W) = 0$, one has $\mathcal{F}^{q+1} W = 0$, if $W = N_q W$.

**Corollary 6.24.** *If $n = \infty$ then $gr^1_\mathcal{F} C_{k(X)} = \mathrm{Alb}X(F)_\mathbb{Q}$ and $C_{k(X)} \cong \mathbb{Q} \oplus \mathrm{Alb}X(F)_\mathbb{Q} \oplus \mathcal{F}^2 C_{k(X)}$ for any smooth proper $k$-variety $X$.*

*Proof.* By Corollary 6.22, $\mathcal{F}^1 W = \bigcap \ker(W \xrightarrow{\varphi} \mathbb{Q} \oplus W')$, where $W'$ runs over all quotients of direct sum of modules $\mathcal{A}(F)_\mathbb{Q}$ for some abelian varieties $\mathcal{A}$ over $k$ by a trivial submodule. We may suppose that $\varphi$'s are surjective. As $C_{k(X)}$ is projective, any such $\varphi$ lifts to a homomorphism to the direct sum of $\mathcal{A}(F)_\mathbb{Q}$.

Let $C^\circ_{k(X)} = \ker(C_{k(X)} \xrightarrow{\deg} \mathbb{Q})$, so $C_{k(X)} \cong \mathbb{Q} \oplus C^\circ_{k(X)}$. One has

$$\begin{aligned}\mathrm{Hom}_G(C^\circ_{k(X)}, \mathcal{A}(F)_\mathbb{Q}) &= \mathrm{Hom}_G(C_{k(X)}, \mathcal{A}(F)_\mathbb{Q})/\mathrm{Hom}_G(\mathbb{Q}, \mathcal{A}(F)_\mathbb{Q}) \\ &= (\mathcal{A}(k(X))/\mathcal{A}(k))_\mathbb{Q} = (\mathrm{Mor}(X, \mathcal{A})/\mathcal{A}(k))_\mathbb{Q} \\ &= \mathrm{Hom}(\mathrm{Alb}X, \mathcal{A})_\mathbb{Q} = \mathrm{Hom}_G(\mathrm{Alb}X(F)_\mathbb{Q}, \mathcal{A}(F)_\mathbb{Q}),\end{aligned}$$

so any $G$-homomorphisms from $C^{\circ}_{k(X)}$ to any object of level 1 factors through $\mathrm{Alb}X(F)_{\mathbb{Q}}$.

The filtration splits, since $\mathrm{Alb}X(F)_{\mathbb{Q}}$ is projective. $\square$

### *6.3 The inner $\mathcal{H}om$*

**Corollary 6.25.** *The inclusion $\mathbb{Q}[\{L(X) \overset{/L}{\hookrightarrow} F\}] \subseteq \mathbb{Q}[\{k(X) \overset{/k}{\hookrightarrow} F\}]$ induces a surjection of $U_L$-modules $\mathbb{Q}[\{L(X) \overset{/L}{\hookrightarrow} F\}] \longrightarrow C_{k(X)}$ for any extension $L$ of $k$ in $F$ with $\mathrm{tr.deg}(F/L) = \infty$ and any irreducible $k$-variety $X$.*

*Proof.* For any $\sigma : k(X) \overset{/k}{\hookrightarrow} F$ there is a generic curve $Y$ on $X_F$ defined over some $k'$ such that $\sigma$ is its $F$-point generic with respect to $k'$. Then the class of $\sigma$ in $\mathcal{I}_{k'}\mathbb{Q}[\{k'(Y) \overset{/k'}{\hookrightarrow} F\}]$ can be presented by a linear combination of $F$-points of $Y$ generic with respect to $k'L$. As generic points of $Y$ are generic points of $X$, this means that $\mathbb{Q}[\{L(X) \overset{/L}{\hookrightarrow} F\}] \longrightarrow C_{k(X)}$ is surjective. $\square$

Propositions 6.15 and the following one suggest that the category $\mathcal{I}_G$ should be related to the category of *effective* homological motives.

**Proposition 6.26.** *The inner $\mathcal{H}om$ functor*[13] *on $\mathcal{S}m_G$ induces the inner $\mathcal{H}om$ functor on $\mathcal{I}_G$ if $n = \infty$. Level of $\mathcal{H}om(W_1, W_2)$ does not exceed $q$ if $W_1, W_2 = N_qW_2 \in \mathcal{I}_G$ and $q \le 1$.*

*Proof.* For any $W_1 \in \mathcal{I}_G$ there is a collection $I$ of irreducible varieties over $k$ and a surjection of $\bigoplus_{X\in I} C_{k(X)}$ onto $W_1$, and thus, an inclusion of $G$-modules

$$\mathcal{H}om(W_1, W_2) \hookrightarrow \prod_{X\in I} \mathcal{H}om(C_{k(X)}, W_2).$$

Clearly, for any collection $\{M_\alpha\}_{\alpha\in I}$ of objects of $\mathcal{I}_G$ any *smooth* submodule in $\prod_{\alpha\in I} M_\alpha$ is also an object of $\mathcal{I}_G$, and thus, to show that $\mathcal{H}om(W_1, W_2) \in \mathcal{I}_G$ for any $W_1, W_2 \in \mathcal{I}_G$ it suffices to check that $\mathcal{H}om(W_1, W_2) \in \mathcal{I}_G$ for any $W_2 \in \mathcal{I}_G$ and any $W_1$ of type $C_{k(X)}$.

By its definition, $\mathcal{H}om(W_1, W_2)$ is the union of $\mathrm{Hom}_{U_L}(W_1, W_2)$ over all open subgroups $U_L$ in $G$.

Fix an algebraically closed subfield $F'$ of $F$ such that $\mathrm{tr.deg}(F/F') = \mathrm{tr.deg}(F'/k) = \infty$ and an embedding $\sigma : k(X) \overset{/k}{\hookrightarrow} F$ in general position with respect to $F'$.

As, by Corollary 6.25, for any $L \subset F'$ with $\mathrm{tr.deg}(F/L) = \infty$ there is a surjection of $U_L$-modules $W := \mathbb{Q}[\{L(X) \overset{/L}{\hookrightarrow} F\}] \longrightarrow W_1$, one has

[13] $(W_1, W_2) \longmapsto \varinjlim_{U} \mathrm{Hom}_U(W_1, W_2)$, where $U$ runs over the set of open subgroups of $G$. Clearly, $\mathrm{Hom}_G(W_1, \mathcal{H}om(W_2, W_3)) = \mathrm{Hom}_G(W_1 \otimes W_2, W_3)$ for any smooth $G$-modules $W_1, W_2, W_3$.

$\mathrm{Hom}_{U_L}(W_1, W_2) \subseteq \mathrm{Hom}_{U_L}(W, W_2) \cong W_2^{U_{L\sigma(k(X))}}$, and thus, the $G_{F'/k}$-module $\mathcal{H}om(W_1, W_2)^{G_{F/F'}}$ embeds into the $G_{F'/k}$-module $W_2^{U_{F'\sigma(k(X))}} =: W_0$. Here the group $G_{F'/k} = G_{F'\sigma(k(X))/\sigma(k(X))}$ acts on $W_0$ as the quotient of $G_{F/\sigma(k(X))}$ by $G_{F/F'\sigma(k(X))}$. Clearly, $W_0 \in \mathcal{I}_{G_{F'/k}}$, so the smooth $G_{F'/k}$module $\mathcal{H}om(W_1, W_2)^{G_{F/F'}}$ belongs to $\mathcal{I}_{G_{F'/k}}$. By Lemma 6.7, the functor $\mathcal{S}m_G \xrightarrow{H^0(G_{F/F'},-)} \mathcal{S}m_{G_{F'/k}}$ is an equivalence of categories, inducing an equivalence of $\mathcal{I}_G$ and $\mathcal{I}_{G_{F'/k}}$, so the object $\mathcal{H}om(W_1, W_2)$ belongs to $\mathcal{I}_G$.

If $W_2$ is a trivial $G$-module then $\mathcal{H}om(C_{k(X)}, W_2) = W_2$, so the representation $\mathcal{H}om(W_1, W_2)$ is a submodule of a trivial $G$-module, and thus, $\mathcal{H}om(W_1, W_2)$ is trivial itself.

To show that $\mathcal{H}om(W_1, W_2)$ is of level 1, if $W_2 = N_1 W_1$, it is sufficient to treat the case $W_1 = \bigoplus_{X \in I} C^\circ_{k(X)}$ and $W_2 = (\bigoplus_{j \in J} \mathcal{A}_j(F)_{\mathbb{Q}})/\Lambda$ for some $\Lambda \subseteq \bigoplus_{j \in J} \mathcal{A}_j(k)_{\mathbb{Q}}$. Then, using Corollaries 6.24 and 6.25, we get that

$$\mathrm{Hom}_{U_L}(\mathbb{Q}[\{L(X) \overset{/L}{\hookrightarrow} F\}]^\circ, W_2) = \mathrm{Hom}_{U_L}(\mathrm{Alb}X(F)_{\mathbb{Q}}, W_2).$$

As this is independent of $L$, the $G$-module $\mathcal{H}om(W_1, W_2)$ is trivial. □

Remark. The $G$-equivariant pairing $\mathbb{Q}[G/U] \otimes \mathbb{Q}[G/U] \longrightarrow \mathbb{Q}$ given by $[\sigma] \otimes [\tau] \longmapsto 0$ if $[\sigma] \neq [\tau]$ and $[\sigma] \otimes [\sigma] \longmapsto 1$ defines an embedding of $\mathbb{Q}[G/U]$ into its contragredient, so, unlike the objects of $\mathcal{I}_G$ in the case $n = \infty$, for any $1 \leq n \leq \infty$ there exist many smooth $G$-modules with non-trivial contragredients.

### *6.4 A tensor structure on $\mathcal{I}_G$*

As Example after Proposition 6.8 on p.44 shows, $\mathcal{I}_G$ is not closed under tensor products in $\mathcal{S}m_G$. We define $W_1 \otimes_{\mathcal{I}} W_2$ by $\mathcal{I}(W_1 \otimes W_2)$.

This operation is not associative on $\mathcal{S}m_G$ as one can see from the following example. Let $W_j = \mathbb{Q}[G/U_j]$ for some open subgroups $U_j = U_{L_j}$ in $G$, $1 \leq j \leq N$, $N \geq 2$. Then one has $W_1 \otimes \cdots \otimes W_N = \mathbb{Q}[\prod_{j=1}^N G/U_j] = \bigoplus_{\tau \in G\backslash(\prod_j G/U_j)} \mathbb{Q}[G \cdot (\tau_1, \tau_2, \ldots, \tau_N)]$.

Clearly, the representation $\mathbb{Q}[G \cdot (\tau_1, \tau_2, \ldots, \tau_N)]$ is isomorphic to the representation $\mathbb{Q}[G/(\bigcap_{j=1}^N \tau_j U_j \tau_j^{-1})]$, so

$$\mathcal{I}(W_1 \otimes \cdots \otimes W_N) \cong \bigoplus_{(\tau_j) \in G\backslash(\prod_j G/U_j)} C_{L_1 \tau_2(L_2) \ldots \tau_N(L_N)}.$$ [14]

[14] More symmetrically, $W_1 \otimes \cdots \otimes W_N \cong \bigoplus_{x \in \mathbf{Spec}(L_1 \otimes_k \cdots \otimes_k L_N)} \mathbb{Q}[\{k(x) \overset{/k}{\hookrightarrow} F\}]$, so $\mathcal{I}(W_1 \otimes \cdots \otimes W_N) \cong \bigoplus_{x \in \mathbf{Spec}(L_1 \otimes_k \cdots \otimes_k L_N)} C_{k(x)}$.

If $U_1 = U_2 = U_{k(x)}$, one has $\mathcal{I}W_1 = \mathcal{I}W_2 = \mathbb{Q}$, and therefore, $W_1 \otimes_{\mathcal{I}} (W_2 \otimes_{\mathcal{I}} \mathbb{Q}) = W_1 \otimes_{\mathcal{I}} \mathcal{I}W_2 = \mathcal{I}W_1 = \mathbb{Q}$.

On the other hand, by Noether normalization, $(W_1 \otimes_{\mathcal{I}} W_2) \otimes_{\mathcal{I}} \mathbb{Q} = \mathcal{I}(W_1 \otimes W_2)$ contains submodules isomorphic to $C_L$ for any field $L$ finitely generated over $k$ and with $\mathrm{tr.deg}(L/k) = 1$.

**Lemma 6.27.** *If $n = \infty$ then for any finite collection of smooth irreducible proper $k$-varieties $X_1, \dots, X_N$ there is a canonical surjection of $G$-modules $C_{k(X_1 \times_k \cdots \times_k X_N)} \xrightarrow{\mathcal{I}(\alpha)} \mathcal{I}\left(C_{k(X_1)} \otimes \cdots \otimes C_{k(X_N)}\right)$.*

*If $C_{k(X_1 \times_k \cdots \times_k X_N)} = CH_0(X_1 \times_k \cdots \times_k X_N)_{\mathbb{Q}}$ then $\mathcal{I}(\alpha)$ is an isomorphism.*

*Proof.* It suffices to check that the canonical $G$-homomorphism $\mathbb{Q}[\{k(X_1) \otimes_k \cdots \otimes_k k(X_N) \overset{/k}{\hookrightarrow} F\}] \xrightarrow{\alpha} C_{k(X_1)} \otimes \cdots \otimes C_{k(X_N)}$, given by $\tau \longmapsto \tau|_{k(X_1)} \otimes \cdots \otimes \tau|_{k(X_N)}$, is surjective, and its kernel is contained in the kernel of $\mathbb{Q}[\{k(X_1) \otimes_k \cdots \otimes_k k(X_N) \overset{/k}{\hookrightarrow} F\}] \longrightarrow C_{k(X_1 \times_k \cdots \times_k X_N)}$ (so that a canonical $G$-homomorphism $C_{k(X_1)} \otimes \cdots \otimes C_{k(X_N)} \longrightarrow C_{k(X_1 \times_k \cdots \times_k X_N)}$ is defined, and the composition

$$C_{k(X_1 \times_k \cdots \times_k X_N)} \xrightarrow{\mathcal{I}(\alpha)} \mathcal{I}\left(C_{k(X_1)} \otimes \cdots \otimes C_{k(X_N)}\right) \longrightarrow C_{k(X_1 \times_k \cdots \times_k X_N)}$$

is the identity).

We have to check that for any collection of generic points $\sigma_j \in X_j$, $1 \le j \le N$, the class $\sigma$ of $\sigma_1 \otimes \cdots \otimes \sigma_N$ in $C_{k(X_1)} \otimes \cdots \otimes C_{k(X_N)}$ is a linear combination of the images of generic points of $X_1 \times_k \cdots \times_k X_N$.

By induction on $j$ we show that $\sigma$ is a linear combination of elements of type $\sigma'_1 \otimes \cdots \otimes \sigma'_N$, where $\sigma'_1, \dots, \sigma'_j$ are in general position. For $j = 1$ there is nothing to prove. If $j > 1$ there is a curve $Y$ on $X_j$ defined over a subfield $k' \subset F$ with $\mathrm{tr.deg}(k'/k) < \infty$ such that $\sigma'_j \in Y(F) - Y(\overline{k'})$. Clearly, for any $G$-module $W$ the canonical $G$-homomorphism $W \longrightarrow \mathcal{I}W$ factors through the $G_{F/\overline{k'}}$-homomorphism $W \longrightarrow \mathcal{I}_{\overline{k'}}W$. Here $\mathcal{I}_{\overline{k'}} : \mathcal{S}m_{G_{F/\overline{k'}}} \longrightarrow \mathcal{I}_{G_{F/\overline{k'}}}$ denotes the same functor as $\mathcal{I}$, but in the context of $G_{F/\overline{k'}}$-modules. The embedding $Y \hookrightarrow (X_j)_{k'}$ induces the $G_{F/\overline{k'}}$-homomorphism $\mathbb{Q}[Y(F)] \longrightarrow \mathbb{Q}[X_j(F)]$, and therefore, using Corollary 6.21, the $G_{F/\overline{k'}}$-homomorphism

$$Z_0(Y_{\overline{k'}}) \oplus \mathrm{Pic}(Y_F)_{\mathbb{Q}} \longrightarrow \bigoplus_{x \in (X_j)_{\overline{k'}}} C_{\overline{k'}(x)}.$$

This implies that the image of $\sigma'_j \in Y(F)$ in $\mathcal{I}\mathbb{Q}[X_j(F)] = \bigoplus_{x \in X_j} C_{k(x)}$, which is equal to $[\sigma'_j] \in C_{k(X_j)}$, coincides with the image of a linear combination of some generic points of $Y$ that are in general position with respect to $\sigma'_1, \dots, \sigma'_{j-1}$, i.e., of some points of $Y(F) - Y(k'')$, where $k'' = \overline{k'\sigma'_1(k(X_1)) \cdots \sigma'_{j-1}(k(X_{j-1}))}$. This completes the induction, so we may suppose that $\sigma_1, \dots, \sigma_j$ are in general position. Then there is some $\tau$ :

$k(X_1)\otimes_k\cdots\otimes_k k(X_N)\overset{/k}{\hookrightarrow}F$ such that $\tau|_{k(X_j)}=\sigma_j$ for all $1\le j\le N$, thus implying that $\alpha$ is surjective.

Besides, the representation $C_{k(X_1)}\otimes\cdots\otimes C_{k(X_N)}$ surjects onto the representation $CH_0((X_1)_F)\otimes\cdots\otimes CH_0((X_N)_F)_{\mathbb{Q}}$, and the latter one surjects onto $CH_0((X_1\times_k\cdots\times_k X_N)_F)_{\mathbb{Q}}=C_{k(X_1\times_k\cdots\times_k X_N)}$. $\square$

**Corollary 6.28.** *If $n=\infty$ and Lemma 6.27 is true then $\otimes_{\mathcal{I}}$ is associative, the class of projective objects in $\mathcal{I}_G$ is stable under $\otimes_{\mathcal{I}}$, and $W_1\otimes_{\mathcal{I}}\cdots\otimes_{\mathcal{I}} W_N=\mathcal{I}(W_1\otimes\cdots\otimes W_N)$.*

*Proof.* By Lemma 6.27, the class of $G$-modules of type $C_L$ is stable under $\otimes_{\mathcal{I}}$, and $\otimes_{\mathcal{I}}$ is associative on this class. As any projective object is a direct summand of a direct sum of $G$-modules of type $C_L$, the same holds for the class of projective objects in $\mathcal{I}_G$, and also $W_1\otimes_{\mathcal{I}}\cdots\otimes_{\mathcal{I}} W_N=\mathcal{I}(W_1\otimes\cdots\otimes W_N)$ for projective $W_1,\dots,W_N$.

Any object $W_j\in\mathcal{I}_G$ is the cokernel of a map $\mathcal{Q}_j\xrightarrow{\alpha_j}\mathcal{P}_j$, where $\mathcal{P}_j$ and $\mathcal{Q}_j$ are direct sums of $G$-modules of type $C_L$. This implies that $W_i\otimes_{\mathcal{I}} W_j$ is the cokernel of $\mathcal{P}_i\otimes_{\mathcal{I}}\mathcal{Q}_j\oplus\mathcal{Q}_i\otimes_{\mathcal{I}}\mathcal{P}_j\xrightarrow{id\otimes\alpha_j+\alpha_i\otimes id}\mathcal{P}_i\otimes_{\mathcal{I}}\mathcal{P}_j$, and in general, $(\dots(W_1\otimes_{\mathcal{I}} W_2)\otimes_{\mathcal{I}}\cdots)\otimes_{\mathcal{I}} W_N$ is the cokernel of

$$\bigoplus_{j=1}^{N}\mathcal{I}(\mathcal{P}_1\otimes\cdots\otimes\mathcal{Q}_j\otimes\cdots\otimes\mathcal{P}_N)\xrightarrow{\sum_j id\otimes\cdots\otimes\alpha_j\otimes\cdots\otimes id}\mathcal{I}(\mathcal{P}_1\otimes\cdots\otimes\mathcal{P}_N).$$

Clearly, this is independent of rearrangements of the brackets. $\square$

Remarks. 1. As it follows from Example on p.44, The form $W_1\otimes W_2\longrightarrow W_1\otimes_{\mathcal{I}} W_2$ can be degenerate. (If $W_1=E(F)_{\mathbb{Q}}$ for an elliptic curve $E$ over $k$, and $W_2=W_1/\Lambda$ for some subspace $0\ne\Lambda\subseteq E(k)_{\mathbb{Q}}$ then $W_1\otimes_{\mathcal{I}} W_1=\mathcal{I}(\bigwedge^2 W_1)$ surjects onto $W_1\otimes_{\mathcal{I}} W_2$ with kernel dominated by $W_1\otimes\Lambda$. As the form is skew-symmetric, when lifted to $W_1\otimes W_1$, its left kernel contains $\Lambda$. $\square$)

2. The functor $E(F)\otimes_{\mathcal{I}}$ is not exact. (Applying it to $\Lambda\hookrightarrow W_1$, we get $W_1\otimes\Lambda\longrightarrow\mathcal{I}(W_1^{\otimes 2})=\mathcal{I}(\bigwedge^2 W_1)$, with the kernel containing $S^2\Lambda$, since $S^2\Lambda$ is in the kernel of the composition $W_1\otimes\Lambda\hookrightarrow W_1^{\otimes 2}\longrightarrow\bigwedge^2 W_1$. $\square$)

In particular, if we denote by $\mathrm{Tor}^{\mathcal{I}}_{\bullet}(W,-)$ the left derivatives of the functor $W\otimes_{\mathcal{I}}$ then $\mathrm{Tor}^{\mathcal{I}}_{\bullet}(W_1,W_2)\ncong\mathrm{Tor}^{\mathcal{I}}_{\bullet}(W_2,W_1)$. (As the functor $W\otimes_{\mathcal{I}}$ is right exact, any projective object of $\mathcal{I}_G$ is acyclic, cf., e.g., [GM], Ch.III, §6.12., so if $\mathrm{Tor}^{\mathcal{I}}_{\bullet}(W_1,W_2)\cong\mathrm{Tor}^{\mathcal{I}}_{\bullet}(W_2,W_1)$ then

$$\mathrm{Tor}^{\mathcal{I}}_j(E(F)_{\mathbb{Q}},W)\cong\mathrm{Tor}^{\mathcal{I}}_j(W,E(F)_{\mathbb{Q}})=0$$

if $j>0$, i.e., the functor $E(F)\otimes_{\mathcal{I}}$ should be exact. $\square$)

## 7 Representations induced from the compact open subgroups

In this section we give an example (Corollary 7.3) of a pair of essentially different open compact subgroups $U$ and $U'$ in $G$ with embeddings of $E$-representations $E[G/U] \hookrightarrow E[G/U']$ and $E[G/U'] \hookrightarrow E[G/U]$ of $G$. This implies that $E[G/U]$ and $E[G/U']$ have the same irreducible subquotients. Proposition 7.4 contains one more example of this phenomenon. However, it seems crucial for these examples that the primitive motives of maximal level of models of $F^U$ and $F^{U'}$ coincide (and trivial).

But first, two general remarks.

Remarks. 1. **Representation of $G/G^\circ$.** Let $U$ be a compact open subgroup in $G$. Then there is a surjection of the $E$-representation $E[G/U]$ of $G$ onto any irreducible $E$-representation of $G$ factorizing through the quotient $G/G^\circ$.

2. **Twists by 1-dimensional representations.** Let $\varphi$ be a homomorphism from $G/G^\circ$ to $\mathbb{Q}^\times$. We consider $E[G/U](\varphi)$ as the same $E$-vector space as $E[G/U]$, but with the $G$-action $[\sigma] \overset{\tau}{\longmapsto} \varphi(\tau)\cdot[\tau\sigma]$. Then $\lambda_\varphi([\sigma]) := \varphi(\sigma)\cdot[\sigma]$ defines an isomorphism of representations $E[G/U] \xrightarrow{\lambda_\varphi} E[G/U](\varphi)$ of $G$.

This implies that for any irreducible $E$-representation $W$ of $G$ the multiplicities of $W$ and $W(\varphi)$ in $E[G/U]$ coincide.

### *7.1 Purely transcendental extensions of quadratic extensions*

**Lemma 7.1.** *Let $U$ and $U'$ be open compact subgroups in $G$ such that $U\bigcap U'$ is of index 2 in $U$: $U = (U\bigcap U')\bigcup\sigma(U\bigcap U')$, and $U'\bigcap\sigma U'\sigma \subseteq U$. Then $U$ is the only right $U$-coset in $UU'$. Equivalently, for any $\sigma_1, \sigma_2 \in G$ if $\sigma_1 UU' = \sigma_2 UU'$ then $\sigma_1 U = \sigma_2 U$.*

*Proof.* Equivalence. If $U$ is the only right $U$-coset in $UU'$ and $\sigma_1 UU' = \sigma_2 UU'$ then $\sigma_2^{-1}\sigma_1 U \subseteq UU'$, so $\sigma_2^{-1}\sigma_1 U = U$, i.e., $\sigma_2^{-1}\sigma_1 \in U$. Conversely, suppose that $\sigma_1 UU' = \sigma_2 UU'$ implies $\sigma_1 U = \sigma_2 U$. Then if $\sigma_1 U \subseteq UU'$ one also has $\sigma_1 UU' \subseteq UU'$ and, by the measure argument, $\sigma_1 UU' = UU'$, and thus, $\sigma_1 U = U$.

Now suppose that $\sigma_1 UU' = \sigma_2 UU'$. As $UU' = U'\bigcup\sigma U'$, one has either $\sigma_1 U' = \sigma_2 U'$ and $\sigma_1\sigma U' = \sigma_2\sigma U'$, or $\sigma_1 U' = \sigma_2\sigma U'$ and $\sigma_1\sigma U' = \sigma_2 U'$. The second case can be reduced to the first one by replacing $\sigma_2$ with $\sigma_2\sigma$ (as this does not change $\sigma_2 U$). Now one has $\sigma_1^{-1}\sigma_2 \in U'\bigcap\sigma U'\sigma \subseteq U$, and thus, $\sigma_1 U = \sigma_2 U$. □

Remark. One obviously has $U\bigcap U' = \sigma(U\bigcap U')\sigma \subseteq U'\bigcap(\sigma U'\sigma)$. Under assumptions of Lemma 7.1, this means $U'\bigcap(\sigma U'\sigma) = U\bigcap U'$.

**Lemma 7.2.** *Let $U$ and $U'$ be some open compact subgroups in $G$ such that $U\bigcap U'$ is of index 2 in $U$: $U = (U\bigcap U')\bigcup\sigma(U\bigcap U')$. Suppose that for any integer $N \geq 1$ and any collection $\tau_1, \dots, \tau_N \in U'\sigma - U$ one has $\tau_1\cdots\tau_N \neq 1$.*

*Then the morphism of $E$-representations $E[G/U] \xrightarrow{[\xi]\mapsto[\xi\sigma]+[\xi]} E[G/U']$ of $G$ is injective.*

*Proof.* First, we check that $U'\bigcap(\sigma U'\sigma) \subseteq U$. If $\tau \in U'\bigcap(\sigma U'\sigma) - U$ then $\tau^{-1} \in U'\bigcap(\sigma U'\sigma) - U$, so $1 = \tau\tau^{-1} \in (U' - U)((\sigma U'\sigma) - U) = (U'\sigma - U)(U'\sigma - U)$, which contradicts our assumption when $N = 2$.

Suppose that $\sum_{j=1}^{M} b_j[\sigma_j]$ is in the kernel, i.e., $\sum_{j=1}^{M} b_j([\sigma_j]+[\sigma_j\sigma]) = 0$, where $b_j \neq 0$, $\sigma_j$ are pairwise distinct as elements of $G/U$ and $M \geq 2$. Then, by Lemma 7.1, $\sigma_i UU' \neq \sigma_j UU'$ for $i \neq j$.

One considers the graph whose vertices are the right $U'$-cosets in the union $\bigcup_{j=1}^{M} \sigma_j UU'$, and whose edges are the sets $\sigma_j UU'$ for all $1 \leq j \leq M$ which join the vertices $\sigma_j U'$ and $\sigma_j\sigma U'$. There are at least 2 edges entering to a given vertex, since otherwise this "vertex" is contained in the support of $\sum_{j=1}^{M} b_j([\sigma_j] + [\sigma_j\sigma])$, so there exists a simple cycle in the graph, say, formed by edges $\sigma_1 UU', \dots, \sigma_s UU'$ for some $s \geq 3$, i.e., the intersection of the subsets $\sigma_i UU'$ and $\sigma_j UU'$ in $G$ is non-empty if and only if $|i - j| \in \{0, 1, s-1\}$.

We may suppose that for any $1 \leq j < s$ one has $\sigma_j UU'\bigcap\sigma_{j+1}UU' = \sigma_j U'$, and $\sigma_1 UU'\bigcap\sigma_s UU' = \sigma_s U'$, and therefore, $\sigma_j U' = \sigma_{j+1}\sigma U'$ for any $1 \leq j < s$, and $\sigma_s U' = \sigma_1\sigma U'$.

Then $\sigma_j^{-1}\sigma_{j+1} \in U'\sigma - U$ for any $1 \leq j < s$, and $\sigma_s^{-1}\sigma_1 \in U'\sigma - U$.

As $(\sigma_1^{-1}\sigma_2)\cdots(\sigma_j^{-1}\sigma_{j+1})\cdots(\sigma_{s-1}^{-1}\sigma_s)(\sigma_s^{-1}\sigma_1) = 1$, we get contradiction. □

**Corollary 7.3.** *Let $2 \leq n < \infty$ and $L'' \subset F$ be a subfield finitely generated over $k$ with* $\mathrm{tr.deg}(F/L'') = 1$. *For some $u \in \sqrt{(L'')^\times} - (L'')^\times$ and some $t \in F$ transcendental over $L''$ set $L = L''(u, T)$, where $T = (2t - u)^2$, and $L' = L''(t)$. Then for $U = U_L$ and $U' = U_{L'}$ there exist embeddings $E[G/U'] \hookrightarrow E[G/U]$ and $E[G/U] \hookrightarrow E[G/U']$.*

*Proof.* One has $U\bigcap U' = U_{L''(t,u)}$, $U = (U\bigcap U')\bigcup(U\bigcap U')\sigma$, where $\sigma t = u - t$ and $\sigma|_{L''(u)} = id$, and $U' = (U\bigcap U')\bigcup(U\bigcap U')\tau$, where $\tau u = -u$ and $\tau|_{L''(t)} = id$. This implies that $U'\sigma - U = (U\bigcap U')\tau\sigma$ and $(\tau\sigma)^2 u = u$, $(\tau\sigma)^2 t = t + 2u$, so for any $N \geq 1$ and any $\tau_1, \dots, \tau_N \in U'\sigma - U$ one has $\tau_1 \cdots \tau_N \neq 1$. Similarly, $U\tau - U' = (U\bigcap U')\sigma\tau = (U'\sigma - U)^{-1}$, so for any $N \geq 1$ and any $\tau_1, \dots, \tau_N \in U\tau - U'$ one has $\tau_1 \cdots \tau_N \neq 1$. It follows from Lemma 7.2 that there exist embeddings $E[G/U'] \hookrightarrow E[G/U]$ and $E[G/U] \hookrightarrow E[G/U']$. □

**Proposition 7.4.** *Fix an odd integer $m \geq 1$, and let $m - 1 \leq n \leq \infty$. Fix a collection $x_1, \dots, x_m$ of elements of $F$ with the only relation $\sum_{j=1}^{m} x_j^d = 1$ over $k$, where $d \in \{m+1, m+2\}$. Set $L'' = k(x_1, \dots, x_m)$ and $L = (L'')^{\langle e_1 e_2^2 \cdots e_m^m\rangle}$, where $e_i x_j = \zeta^{\delta_{ij}} \cdot x_j$ for a primitive $d^{\text{th}}$ root of unity $\zeta$. Let $L'$ be a maximal purely transcendental extension of $k$ in $L$. Then for $U = U_L$ and $U' = U_{L'}$ the $E$-representations $E[G/U]$ and $E[G/U']$ of $G$ have the same irreducible subquotients.*

*Proof.* As $E[G/U']$ embeds naturally into $E[G/U]$, any subquotient of the module $E[G/U']$ is a subquotient of $E[G/U]$. Let $W$ be the quotient of $E[G/U_{L''}]$ by the sum of the images of $E[G/U']$ under all possible $E[G]$-homomorphisms to $E[G/U_{L''}]$. As in the proof of $CH_0(Y_{[L]}) = \mathbb{Z}$,[15] one checks that $W^{\langle e_1 e_2^2 \cdots e_m^m \rangle} = 0$, and thus, $E[G/U]$ coincides with the sum of the images of $E[G/U']$ under all $E[G]$-homomorphisms to $E[G/U]$. □

REMARK. Let $L$ be an extension of $k$ of finite type and of transcendence degree $q$ in $F$. Then, at least assuming some conjectures, any motivic $G$-module of level $< q$ is a subquotient of $\mathbb{Q}[G/U_L]$ with infinite multiplicity. To see this, fix a transcendence basis $x_1, \dots, x_q$ of $L$ over $k$. Then there is a surjection $\mathbb{Q}[G/U_L] \longrightarrow \Omega^s_{F/k}$, given by $[1] \longmapsto x_{s+1} dx_1 \wedge \cdots \wedge dx_s$ for any $s < q$. Any motivic $G$-module of level $s$ is a submodule of $\Omega^s_{F/k}$ with infinite multiplicity.

## A The centers of the Hecke algebras

**Lemma A.1.** *Let $K$ be a compact open subgroup in $G$. Let $\nu \in \mathcal{H}_E(K)$ be an element which is not a $E$-multiple of $h_K$. Then there exist elements $x_1, \dots, x_n \in F^K$ algebraically independent over $k$ such that $\nu h_U \notin E \cdot h_U$, where $U = U_{k(x_1,\dots,x_n)} \supseteq K$.*

*Proof.* Let $\nu = \sum a_j \sigma_j h_K$, where the classes of $\sigma_j$ in $G/K$ are pairwise distinct. After subtracting a multiple of $h_K$, if necessary, we may suppose that $\sigma_j \notin K$ for any $j$. Then the sets $\{x \in F^K \mid \sigma_i x = \sigma_j x\}$ for $i \neq j$ and $\{x \in F^K \mid \sigma_j x = x\}$ for any $j$ are proper $k$-subspaces in $F^K$, and therefore, there exist elements $x_1, \dots, x_n \in F^K$ algebraically independent over $k$ with $x_1$ outside their union. These conditions on $x_1, \dots, x_n$ imply that $\sigma_i|_{k(x_1,\dots,x_n)} \neq \sigma_j|_{k(x_1,\dots,x_n)}$ and $\sigma_j|_{k(x_1,\dots,x_n)} \neq id$ for any $i \neq j$.

Set $U = U_{k(x_1,\dots,x_n)}$. Then the support of the element $\nu * h_U$ coincides with $\bigcup_j \sigma_j U$, which is not a subset in $U$, so $\nu * h_U$ is not a multiple of $h_U$. □

**Lemma A.2.** *Let $U = U_{k(x_1,\dots,x_n)}$ for some $x_1, \dots, x_n$ algebraically independent over $k$, and let $\nu$ be a central element either in the Hecke algebra $\mathcal{H}_E(U)$, or in the Hecke algebra $\mathcal{H}^\circ_E(U)$. Then $\nu \in E \cdot h_U$.*

*Proof.* For any $\tau$ in the normalizer of $U$ one has $\nu(h_U \tau h_U) = \nu h_U \tau = \nu \tau \neq 0$ if $\nu \neq 0$, and $(h_U \tau h_U)\nu = \tau h_U \nu = \tau \nu$. We may suppose that the support of $\nu$ does not contain 1, i.e., $\mathrm{Supp}(\nu) = \coprod_{\sigma \in S} U \sigma U$ for a finite subset $S$ in $G - U$.

[15] Let $A$ be the image of $\mathbb{Q}[e_1, e_2, \dots, e_m]$ in $\mathrm{End}_G W$. It is a semisimple algebra, so we want to show that $e_1 e_2^2 \cdots e_m^m \not\equiv 1$ modulo any maximal ideal in $A$. For this we note that $(L'')^{\langle e_{i_1} \cdots e_{i_l} \rangle}$ is rational for any $1 \leq i_1 < \cdots < i_l \leq m$, so $\sum_{j=0}^{d-1} (e_{i_1} \cdots e_{i_l})^j = 0$. The assumptions on $m$ and $d$ imply that modulo any maximal ideal in $A$ the element $e_1 e_2^2 \cdots e_m^m$ is a non-trivial root of unity.

Let $H = \{\tau \in G \mid \tau|_{k(x_j)} \in \mathrm{Aut}(k(x_j)/k)$ for all $1 \leq j \leq n\}$, and for each $1 \leq j \leq n$ let the subfield $L_j$ be generated over $k$ by $x_1, \dots, \widehat{x_j}, \dots, x_n$. As $\nu$ is a central element in $\mathcal{H}(U)$, one has $\tau\nu\tau^{-1} = \nu$ for all $\tau \in H$. In particular, $\mathrm{Supp}(\tau\nu\tau^{-1}) = \mathrm{Supp}(\nu)$, so each $\tau \in H$ induces a permutation of the set $S$ of double $U$-classes. The subgroup $U \subset H$ acts trivially on $S$, so the action of $H$ on $S$ factors through the quotient $H/U \cong (\mathrm{PGL}_2 k)^n$. Any homomorphism from $(\mathrm{PGL}_2 k)^n$ to the permutation group of the set $S$, is trivial, since any element of $\mathrm{PGL}_2 k$ is $(\#S)$!-th power of another element of $\mathrm{PGL}_2 k$, and therefore, $U\tau\sigma\tau^{-1}U = U\sigma U$ for any $\sigma \in S$. In particular, $\tau\sigma\tau^{-1}x_j$ is in the finite set $U\sigma x_j$ for all $\tau \in H$; or, even more particularly, the set of fields $k(\tau\sigma x_j)$ for all $\tau \in H$ is finite.

Fix some $j$. Suppose that $\sigma x_j \notin \overline{k(x_j)}$ (this implies that $n > 1$). Then there is $1 \leq s \leq n$ different from $j$ such that $F$ is algebraic over $L_s(\sigma x_j)$. Set $H_j = \{\tau \in U_{L_s} \mid \tau|_{k(x_s)} \in \mathrm{Aut}(k(x_s)/k)\}$. Then for any $\tau \in H_j$ one has $\tau\sigma\tau^{-1}x_j = \tau\sigma x_j$, so the $H_j$-orbit of $\sigma x_j$ should be finite, and thus, a subgroup of finite index in $H_j$ should be compact, so the group $H_j$ should be compact itself, which is false.

As $U\tau\sigma\tau^{-1}U = U\sigma U$ is equivalent to $U\tau\sigma^{-1}\tau^{-1}U = U\sigma^{-1}U$, we get $\sigma^{\pm 1}x_j \in \overline{k(x_j)}$. If $\sigma^{\pm 1}x_j \notin k(x_j)$ then $k(\sigma^{\pm 1}x_j, x_j)/k(x_j)$ has a non-empty branch locus. The $\mathrm{PGL}_2 k$-orbit of any point on $\mathbb{P}^1_k$ is infinite, so the $\mathrm{PGL}_2 k$-orbit of the branch locus is also infinite, which means that the set of fields $k(\tau\sigma^{\pm 1}x_j)$ is infinite, unless $k(\sigma^{\pm 1}x_j)$ is a subfield in $k(x_j)$. Then $k(x_j) = \sigma k(\sigma^{-1}x_j) \subseteq \sigma k(x_j) = k(\sigma x_j) \subseteq k(x_j)$. As the center of $\mathrm{PGL}_2 k$ is trivial, this shows that $\sigma|_{k(x_j)} = id$. When varying $j$, we get $\sigma \in U$, contradicting our assumptions. $\square$

**Lemma A.3.** *Let $K$ be a compact subgroup in $G$. If $n < \infty$ and $\nu \in \mathcal{H}_E(K) - E \cdot h_K$ then there exists a compact open subgroup $U$ containing $K$ such that $\nu * h_U \notin E \cdot h_U$.*

*Proof.* There is some $\sigma$ in the support of $\nu$ outside of $K$, i.e., if $U'$ is an open compact subgroup in $G$ not containing $\sigma$ then there is an open subgroup $U \subseteq U'$ such that $\nu(\sigma U) \neq 0$, and therefore, the support of $\nu h_U$ contains $\sigma$, so it is non-empty and it does not coincide with $U$. $\square$

As a corollary of these statements we get

**Theorem A.4.** *Let $K$ be a compact subgroup in $G$. Then the centers of the Hecke algebras $\mathcal{H}_E(K)$ and $\mathcal{H}^\circ_E(K)$ coincide with $E \cdot h_K$ if $n < \infty$.*

*Proof.* Clearly, for any pair of compact subgroups $K \subseteq U$ the multiplication by $h_U$: $\nu \longmapsto \nu h_U$ gives homomorphisms of the centers $Z(\mathcal{H}_E(K)) \xrightarrow{h_U *} Z(\mathcal{H}_E(U))$ and $Z(\mathcal{H}^\circ_E(K)) \xrightarrow{h_U *} Z(\mathcal{H}^\circ_E(U))$. Then by Lemma A.3, we may suppose that $K$ is open. By Lemma A.1, we may further suppose that $K = U_{k(x_1,\dots,x_n)}$. Then, by Lemma A.2, the centers of $\mathcal{H}_E(K)$ and $\mathcal{H}^\circ_E(K)$ coincide with $E \cdot h_K$. $\square$

## B The case of positive characteristic

In this appendix we show that all results of §2 and §6 remain valid in the case of $\mathrm{char}(k) = p > 0$.

The topology on $G$ is the same as described in Introduction. The group $G$ is also Hausdorff, locally compact if $n < \infty$, and totally disconnected; the subgroups $G_{\{F,(F_\alpha)_{\alpha\in I}\}/k}$ are closed in $G$, the fibers of the morphism of unitary semigroups

$$\{\text{subfields in } F \text{ over } k\} \longrightarrow \{\text{closed subgroups in } G\}$$

given by $K \longmapsto \mathrm{Aut}(F/K)$ consist of subfields of $F$ with the same sets of perfect subfields containing them (with the same "perfectization"), its image is stable under passages to sub-/sup-groups with compact quotients, and it induces bijections

– $\left\{\begin{array}{c}\text{perfect subfields } K \subset F\\ \text{over } k \text{ with } F = \overline{K}\end{array}\right\} \leftrightarrow \{\text{compact subgroups of } G\}$;

– $\left\{\begin{array}{c}\text{perfect subfields } K \text{ of } F \text{ minimal over}\\ \text{subfields of finite type over } k \text{ with } F = \overline{K}\end{array}\right\} \leftrightarrow \left\{\begin{array}{c}\text{compact open}\\ \text{subgroups of } G\end{array}\right\}$.

The inverse correspondences are given by $G \supset H \longmapsto F^H$.

*Proof of Lemma 2.1.* We may suppose that $L$ and all $L_\alpha$ are perfect. Then, after replacing the reference to the Galois correspondence of §2 with the reference to the Galois correspondence of this appendix, the proof in §2 goes through. $\square$

*Proof of Lemma 2.2.* First, replace $L$ with its "perfectization". Let $\ell \in \{2,3\} - \{p\}$. Then any element $\tau$ in the common normalizer in $G$ of all closed subgroups of index $\le \ell$ in $U_L$ satisfies $\tau(L(f^{1/\ell})) = L(f^{1/\ell})$ for all $f \in L^\times$. If $\tau \notin U_L$ then there is an element $x \in L^\times$ such that $\tau x/x \neq 1$. Then $\tau x/x = y^\ell$ for some $y \in F^\times - \mu_\ell$. Set $f = x + \lambda$ for a variable $\lambda \in k$. By Kummer theory, $\tau f/f \in L^{\times\ell}$, and therefore, $L$ contains $L_0 := k\left(y\frac{(x+\lambda y^{-\ell})^{1/\ell}}{(x+\lambda)^{1/\ell}} \mid \lambda \in k\right) \subset \overline{k(x,y)}$.

Now we come back to our original $L$ and replace $L_0$ with the subfield generated by appropriate $p$-primary powers of $y\frac{(x+\lambda y^{-\ell})^{1/\ell}}{(x+\lambda)^{1/\ell}}$, where $p = \mathrm{char}(k)$. As $\mathrm{tr.deg}(\overline{k(x,y)}/k) \le 2$, by our assumption on $L$, the subfield $L_0$ of $L$ should be finitely generated over $k$. But this is possible only if $y^\ell = 1$, i.e., if $\tau \in U_L$. (To see this, one can choose a smooth model of the extension $L_0(x)/k(x,y)$ over $k$ and look at its branch locus.) $\square$

*Proof of Lemma 2.4.* Let $\sigma \in H \bigcap U - \{1\}$ and $k'$ be the algebraic closure in $F$ of any subfield in $F^{\langle\sigma\rangle}$ with $\mathrm{tr.deg}(F/k') = 1$. As the extension $F/F^{\langle\sigma\rangle}$ is abelian there is an element $x \in F - k'$ and an integer $N \ge 2$ such that $\sigma x \neq x$ and either $\sigma x^N = x^N$, or $\sigma(x^p - x) = x^p - x$. Then one has $\sigma(k') = k'$ and $\sigma(k'(x)) = k'(x)$.

The rest of the proof is the same as the last two paragraphs of the proof in §2. $\square$

*Proof of Lemma 2.5* in §2 remains valid, after we replace $L'$ with its perfect closure in $LL'$, but we do not claim that the inclusion $\mathrm{PGL}_2k \hookrightarrow N_{G^\circ}U_{L'}/U_{L'}$ is an isomorphism. □

*Proof of Lemma 2.7.* We proceed by induction on $m$, the case $m = 0$ being trivial. We wish to find $w_m \in F$ such that $w$ and $\xi w_m$ are algebraically independent over $k'$ generated over $k$ by $w_1, \dots, w_{m-1}, \xi w_1, \dots, \xi w_{m-1}$. Suppose that there is no such $w_m$. Then for any $u \in F - \overline{k'}$ and any $v \in F - \overline{k'(\xi u)}$ one has the following vanishings in $\Omega^2_{k'(u,v,\xi u,\xi v)/k'}$: $du \wedge d\xi u = dv \wedge d\xi v = 0$, $d(u+v) \wedge d\xi(u+v) = 0$, and $d(u+v^\ell) \wedge d\xi(u+v^\ell) = 0$ for any integer $\ell \geq 2$ prime to $p := \mathrm{char}(k)$. Applying the first two to the third, we get $\ell(v^{\ell-1} - \xi v^{\ell-1})dv \wedge d\xi u = 0$, which means that either $\xi v^{\ell-1} = v^{\ell-1}$ for any $v \in F - \overline{k'(\xi u)}$, or $dv = 0 \in \Omega^1_{k'(v,\xi v)/k'}$ for all $v \in F - \overline{k'}$, or $d\xi u = 0 \in \Omega^1_{k'(u,\xi u)/k'}$ for all $u \in F - \overline{k'}$. In the first case $\xi v = v$ for any $v \in F$, i.e., $\xi = 1$.

If $\xi(\overline{k'}) \neq \overline{k'}$ then there exists $u \in \overline{k'}$ such that $\xi u \in F - \overline{k'}$. Fix some $v \in F - (\overline{k'(\xi u)} \cup \xi^{-1}(\overline{k'}))$. Even if $\xi v \in \overline{k'(v)}$, the element $\xi(uv)$ does not belong to $\overline{k'(v)} = \overline{k'(uv)}$, i.e., $\xi(uv)$ and $uv$ are algebraically independent over $k'$.

We may, thus, suppose that $\xi(\overline{k'}) = \overline{k'}$. Replacing $\xi$ with $\xi^{-1}$, we reduce the case $d\xi u = 0 \in \Omega^1_{k'(u,\xi u)/k'}$ to the case $du = 0 \in \Omega^1_{k'(u,\xi u)/k'}$ for all $u \in F - \overline{k'}$. Let $P(X, Y^{p^s})$ be the minimal polynomial of $\xi u$ over $\overline{k'}[u]$ with maximal possible integer $s \geq 1$.

Replacing $\xi$ by $\mathrm{Fr}^s\xi$, where Fr is the Frobenius automorphism, we get that $du \neq 0 \in \Omega^1_{k'(u,\xi u)/k'}$. As this implies $\xi = 1$, we get contradiction, since no non-zero power of the Frobenius automorphism is identical on $k$. This shows that there exists desired $w_m \in F$. □

*Proof of Proposition 2.14.* We may suppose that $L_1$ and $L_2$ are perfect. Then the proof in §2 goes through. □

Proofs of Lemmas 2.6, 2.8, 2.12, 2.15 and 2.16, Theorem 2.9, and Corollaries 2.3, 2.10, 2.11 and 2.13 given in §2 remain valid without any changes.

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